Stabilization of nonlinear sandwich systems via state feedback – Discrete-time systems

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SUMMARY

A recent paper [1] considered stabilization of a class of continuous-time nonlinear sandwich systems via state feedback. This paper is a discrete-time counterpart of it. The class of nonlinear sandwich systems consists of saturation elements sandwiched between linear systems. We focus first on single-layer sandwich systems, which consist of a single saturation sandwiched between two linear systems. For such systems, we present necessary and sufficient conditions for semi-global and global stabilization by state feedback, and develop design methodologies to achieve the prescribed stabilization. We extend the results to single-layer sandwich systems subject to additional actuator saturation. Finally, we discuss further extension to general multi-layer sandwich systems with an arbitrary number of saturations sandwiched between linear systems, both with and without actuator saturation. The design methodologies can be viewed as extensions of classical low-gain design methodologies developed during 1990's in the context of stabilizing linear systems subject to actuator saturation.

1. Introduction

Physical systems are typically made up of interconnected subsystems, some of which are well-characterized as linear, and some of which are distinctly nonlinear. Many systems can therefore be described as an interconnection of separable linear and nonlinear parts. A common type of structure consists of a static nonlinearity *sandwiched* between two linear systems. One of the ubiquitous static

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nonlinearities is a saturation element. We observe that the resulting sandwiched non-linear systems, as shown in Figure 1 are extensive generalizations of linear systems subject to actuator saturation.

A recent paper [1] focuses on continuous-time sandwich systems where the static nonlinearity is a saturation. This paper is a counterpart of the same problem, for discrete-time systems. As one can expect, some aspects of development for discrete-time systems are analogous to those in continuous-time systems. On the other hand, some other aspects are distinctly different and require subtle and important changes. Hence, for continuity and readability, this paper is written independent of its continuous-time version.

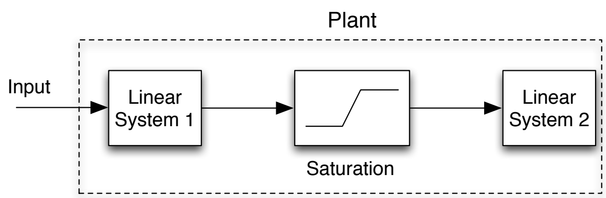

Figure 1. Single-layer sandwich system

Figure 1 depicts a single-layer sandwich system, where the single layer refers to the saturation element that is sandwiched between two linear systems. A natural extension of this class of systems is depicted in Figure 2, which shows a single-layer sandwich system subject to actuator saturation. These types of systems can be further extended to multi-layer sandwich systems, and multi-layer sandwich systems subject to actuator saturation, shown in Figures 3 and 4.

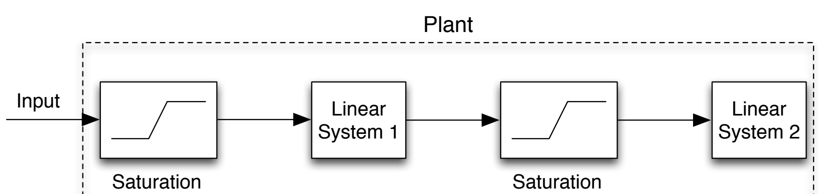

Figure 2. Single-layer sandwich system subject to actuator saturation

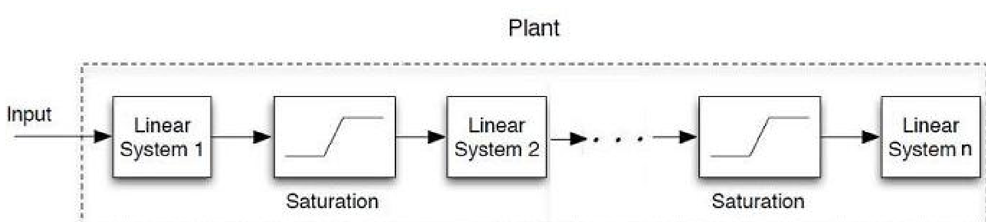

Figure 3. Multi-layer sandwich system

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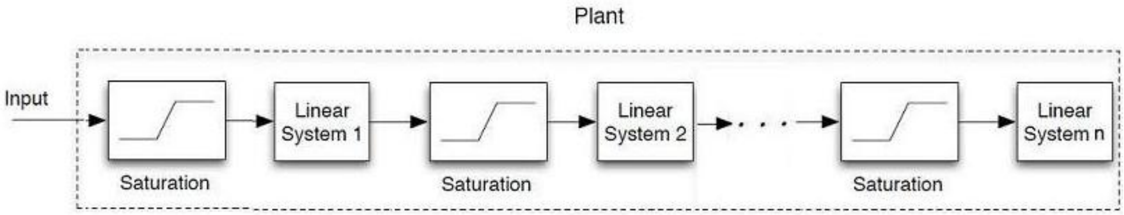

Figure 4. Multi-layer sandwich system subject to actuator saturation

1.1. Previous work

Sandwiched systems such as those depicted in Figure 1 are a special case of so-called cascaded systems which are linear systems whose output affects a nonlinear system. This research was initiated in [2] but has also been studied for instance in [3, 4]. Note that in our case the nonlinear system has a very special structure of an interconnection of a static nonlinearity with a linear system. Moreover, in these references the nonlinear system is assumed to be stable and the goal was to see whether the output of a stable linear system can affect this stability. The goal of this paper is focused on stabilization and design of controllers and is inherently different.

Some researchers have previously studied linear systems with sandwiched nonlinearities. The most recent activity in this area is the work of Tao et al. [5, 6, 7, 8]. The main technique used in these papers is based on an approximate inversion of the nonlinearities. An example studied in these references is a deadzone, which is a right-invertible nonlinearity. By contrast, a saturation has a very limited range and cannot be inverted even approximately, except in a local region. The work of Tao et al. is therefore not applicable to the case of a saturation nonlinearity. To achieve our goal of semi-global and global stabilization, we need to face the saturation directly, by exploiting the structural properties of the given linear systems.

The systems illustrated by Figures 1–4 are progressive generalizations of the class of systems consisting of a single linear system with an actuator saturation. Over the past years there has been a strong interest in stabilization of this class of systems. Several important results have appeared in the literature, starting with the works of Fuller [9, 10], Sontag and Sussmann [11], Sussmann and Yang [12], as well as Sussmann, Sontag, and Yang [13]. (See also two special issues of the International Journal of Robust and Nonlinear Control [14, 15].) These works led to the development of *low-gain* design methodologies for semi-global stabilization, and *scheduled low-gain* design methodologies for global stabilization of linear systems subject to actuator saturation [16, 17, 18]. Since being developed in the 1990's, low-gain and scheduled low-gain design methodologies have formed an integral part of several related design methodologies, such as low-and-high-gain design methodologies. The scheduled low-gain design methodology is based on the concept of scheduling, developed by Megretski [19].

Recent research has also focused on linear systems subject to state constraints, where the controller must guarantee that the output of a linear system remains in a given set (see, for instance, [20, 21, 22, 23] and references therein). The approach developed in these works can be used for the class of nonlinear sandwich systems, albeit with some drawbacks. First, the approach does not allow arbitrary initial conditions. Instead, the initial conditions must belong to a constrained set known as the *admissible set of initial conditions* to avoid constraints violation at time 0. Second, the approach is based on limiting the input to avoid activation of *all* the saturations for *all* time, so that the closed-

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loop system only operates in the linear region. This requires either further restrictions on the initial conditions or structural constraints on the zeros to be imposed. In our problem formulations, we allow that the saturation is activated and in this way we avoid the above restrictions.

1.2. Contributions of this paper

In this paper, we first establish conditions for semi-global and global stabilizability of single-layer sandwich systems, portrayed in Figure 1, and we construct appropriate control laws by state feedback. We then extend the stabilization results to single-layer sandwich systems subject to actuator saturation, portrayed in Figure 2. The design methodologies that emerge from this extension are generalizations of the classical low-gain and scheduled low-gain design methodologies, developed for semi-global and global stabilization of linear systems subject only to actuator saturation. Indeed, when the first linear system is a static and invertible, the new design methodologies reduce to their classical counterparts, and we therefore refer to the new design methodologies as *generalized low-gain* design (for semi-global stabilization) and *generalized scheduled low-gain* design (for global stabilization). We furthermore discuss the natural extension of the results to the multi-layer sandwich systems portrayed in Figures 3 and 4. We illustrate the results with an example.

2. Problem formulations and preliminaries

2.1. Problem formulation

In this section, we first describe the dynamic equations of the class of single-layer sandwich systems, portrayed in Figure 1. We then formulate the semi-global and global stabilization problems for this class of systems.

Single-layer sandwich systems consist of two interconnected systems, L_1 and L_2, given by

$$L_1 : \begin{cases} x(k+1) = Ax(k) + Bu(k) \\ z(k) \quad = Cx(k), \end{cases} \tag{1}$$

and

$$L_2 : \quad \omega(k+1) = M\omega(k) + N\sigma(z(k)) \tag{2}$$

where $x(k) \in \mathbb{R}^{n_1}$, $\omega(k) \in \mathbb{R}^{n_2}$, $u(k) \in \mathbb{R}^{m_1}$ and $z(k) \in \mathbb{R}^{m_2}$.

As will become clear in the design procedure, different saturation levels do not cause any intrinsic differences in controller design methodology except for some changes on ranges of certain design parameters. Therefore, without loss of generality, we assume that all the saturation elements studied in this paper are indeed the same and equal to the standard saturation function defined as $\sigma(z) = [\sigma_1(z_1), \ldots, \sigma_1(z_{m_2})]'$ where $\sigma_1(s) = \text{sgn}(s)\min\{|s|, \Delta\}$ for some $\Delta > 0$.

The dynamics of system L_1 can be modified to include an actuator saturation, and we refer to the resulting system as $\bar{L}_1$. Single-layer sandwich systems subject to actuator saturation therefore consist of two systems, $\bar{L}_1$ and L_2, given by

$$\bar{L}_1 : \begin{cases} x(k+1) = Ax(k) + B\sigma(u(k)), \\ z(k) \quad = Cx(k), \end{cases} \tag{3}$$

and

$$L_2 : \quad \omega(k+1) = M\omega(k) + N\sigma(z(k)), \tag{4}$$

where $x(k) \in \mathbb{R}^{n_1}$, $\omega(k) \in \mathbb{R}^{n_2}$, $u(k) \in \mathbb{R}^{m_1}$ and $z(k) \in \mathbb{R}^{m_2}$.

This type of system configuration can be generalized to an interconnection of n linear systems, namely the multi-layer nonlinear sandwich systems. Consider the following interconnection of n systems:

$$L_i : \begin{cases} x_i(k+1) = A_i x_i(k) + B_i \sigma(u_i(k)), \quad i = 1, \ldots, n \\ z_i(k) \quad = C_i x_i(k), \quad i = 1, \ldots, n-1 \\ u_i(k) \quad = z_{i-1}(k), \quad i = 2, \ldots, n \end{cases} \tag{5}$$

where $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$ for $i = 1, \ldots, n$, $z_i \in \mathbb{R}^{m_{i+1}}$ for $i = 1, \ldots, n-1$.

Let $\bar{x}$ and u denote the state and input of the over-all sandwich systems. The semi-global and global stabilization problems for the three sandwich systems as defined above can be formulated as follows:

Problem 1. *Consider the (single layer, single layer with input saturation and multilayer) sandwich nonlinear systems as defined above. The* semi-global stabilization problem for sandwich nonlinear systems *is said to be solvable if for any compact subset $\mathscr{W}$ of whole state space, there exists a state feedback control law $u = f(\bar{x})$ such that the origin of the closed-loop system is asymptotically stable with $\mathscr{W}$ contained in its domain of attraction.*

Problem 2. *Consider the (single layer, single layer with input saturation and multilayer) sandwich nonlinear systems as defined above. The* global stabilization problem for the sandwich systems *is said to be solvable if there exists a state feedback control law $u = f(\bar{x})$ such that the origin of the closed-loop system is globally asymptotically stable.*

2.2. Low-gain state feedback design methodology

In this subsection, we recall the low-gain feedback design methodology for stabilization of discrete-time system subject to input saturation.

Consider the system

$$x(k+1) = Ax(k) + B\sigma(u(k)) \tag{6}$$

where $x \in \mathbb{R}^{n_0}$ and $u \in \mathbb{R}^{m_0}$. We have the following lemma:

Lemma 1. *Assume that (A, B) is stabilizable and A has all its eigenvalues inside the closed unit disc. Then the discrete-time algebraic Riccati equation*

$$P_\varepsilon = A' P_\varepsilon A + \varepsilon I - A' P_\varepsilon B (B' P_\varepsilon B + I)^{-1} B' P_\varepsilon A \tag{7}$$

with $\varepsilon \in (0, 1]$ has a unique positive definite solution P_ε. Moreover, this P_ε has the following properties:

1. *For any $\varepsilon \in (0, 1]$, P_ε is such that $A - B(B' P_\varepsilon B + I)^{-1} B' PA$ is asymptotically stable.*
2. $\lim_{\varepsilon \to 0} P_\varepsilon = 0$.
3. *For all $\varepsilon \in (0, 1]$, there exists a $M_p > 0$ such that $\| [P_\varepsilon^{\frac{1}{2}} A P_\varepsilon^{-\frac{1}{2}} \| \leq \sqrt{M_p}$ where*

$$M_p = \sigma_{\max}(P_1^{\frac{1}{2}} B B' P_1^{\frac{1}{2}}) + 1 \tag{8}$$

and P_1 is the solution of ARE (7) with $\varepsilon = 1$.

Proof : Proof of properties 1 and 2 can be found in [24]. Property 3 is a direct result of Lemma 3.2 in [24]. ∎

The low-gain state feedback developed in [24] is a family of parameterized state feedback laws given by

$$u_L(x) = F_\varepsilon x = -(B'P_\varepsilon B + I)^{-1} B'P_\varepsilon Ax \tag{9}$$

where P_ε is the solution of (7). ε is called the low-gain parameter. Lemma 1 implies that the magnitude of the control input can be made arbitrarily small by choosing sufficiently small ε. This implies that in the case of input saturation and initial conditions in a compact set, we can avoid activating the actuator saturation. It has been shown in [24] that this family of parameterized controllers solves the semi-global stabilization problem for a discrete-time system with input saturation described by (6).

2.3. Scheduling of low-gain parameter

In the semi-global framework, with controller (9), the domain of attraction of the closed-loop system is determined by a low-gain parameter ε, which is chosen according to any given a priori set of initial conditions. In order to solve the global stabilization problem, the parameter ε can be scheduled with respect to the state. This has been done in the literature [19].

We are looking for a scheduled parameter satisfying the following properties:

1. $\varepsilon(x) : \mathbb{R}^{n_0} \to (0,1]$ is continuous and piecewise continuously differentiable.
2. There exists an open neighborhood $\mathscr{O}$ of the origin such that $\varepsilon(x) = 1$ for all $x \in \mathscr{O}$.
3. For any $x \in \mathbb{R}^{n_0}$, we have $\|F_{\varepsilon(x)} x\| \leq \delta$ where δ is a design parameter to be selected later.
4. $\varepsilon(x) \to 0$ as $\|x\| \to \infty$.
5. $\{ x \in \mathbb{R}^{n_0} \mid x'P_{\varepsilon(x)} x \leq c \}$ is a bounded set for all $c > 0$.

We basically choose the same scheduling as used in [25]:

$$\varepsilon(x) = \max \{ r \in (0,1] \mid (x'P_r x)\|B'P_r B\| \leq \tfrac{\delta^2}{M_p} \} \tag{10}$$

where P_r is the unique positive definite solution of algebraic Riccati equation (7) with $\varepsilon = r$ and M_p is given by Lemma 1. Properties 1,2,4 and 5 follow directly from continuous-time result. Property 3 follows from the fact that

$$\begin{aligned}
&\|(B'P_{\varepsilon(x)}B + I)^{-1} B'P_{\varepsilon(x)}Ax\| \\
&\leq \|(B'P_{\varepsilon(x)}B + I)^{-1}\| \left\| B'P_{\varepsilon(x)}^{\frac{1}{2}} \right\| \left\| P_{\varepsilon(x)}^{\frac{1}{2}} A P_{\varepsilon(x)}^{-\frac{1}{2}} \right\| \left\| P_{\varepsilon(x)}^{\frac{1}{2}} x \right\| \\
&\leq \sqrt{x'P_{\varepsilon(x)}x} \left\| B'P_{\varepsilon(x)}^{\frac{1}{2}} \right\| \left\| P_{\varepsilon(x)}^{\frac{1}{2}} A P_{\varepsilon(x)}^{-\frac{1}{2}} \right\|.
\end{aligned}$$

Using Lemma 1 we find

$$\|(B'P_{\varepsilon(x)}B + I)^{-1} B'P_{\varepsilon(x)}Ax\| \leq \sqrt{M_p x'P_{\varepsilon(x)}x \|(B'P_{\varepsilon(x)}B)\|} \leq \delta.$$

A family of low-gain feedback controllers for global stabilization is given by

$$u_L(x) = F_{\varepsilon(x)} x = -(B'P_{\varepsilon(x)}B + I)^{-1} B'P_{\varepsilon(x)}Ax.$$

It has been shown in [25], that this scheduled controller (9) with scheduling parameter ε defined by (10) achieves global asymptotic stability of the origin of system (6).

In subsequent sections, we generalize the low-gain feedback design technology to solve the stabilization problems as defined in Problems 1 and 2 for three types of discrete-time nonlinear sandwich systems.

3. Necessary and sufficient conditions for stabilization of nonlinear sandwich systems

3.1. Single layer sandwich system

In this subsection, we present two theorems that give necessary and sufficient conditions for the solvability of semi-global and global stabilization problems as defined in Problems 1 and 2 for a single layer nonlinear sandwich system.

Theorem 1. *Consider the interconnection of the two systems given by* (1) *and* (2). *Both the semi-global and the global stabilization problems, as formulated in Problems 1 and 2 respectively, are solvable if and only if,*

1. *The linearized cascaded system is stabilizable, i.e.* $(\mathscr{A},\mathscr{B})$ *is stabilizable, where*

$$\mathscr{A} = \begin{pmatrix} A & 0 \\ NC & M \end{pmatrix} \text{ and } \mathscr{B} = \begin{pmatrix} B \\ 0 \end{pmatrix}. \tag{11}$$

2. *All the eigenvalues of M are in the closed unit disc.*

Proof : Necessity of both the conditions is quite immediate. The system L_2 needs to be stabilized through a saturated signal and it is well known, see for instance [13], that this can only be done if the eigenvalues of M are in the closed unit disc. The cascaded system is linear in a small neighborhood around $(0,0)$ and hence the stabilizability of the nonlinear cascaded system clearly requires the stabilizability of the local linear system, which is equivalent to the stabilizability of the pair $(\mathscr{A},\mathscr{B})$.

Sufficiency is established in the next section by an explicit construction of a stabilizing controller. ∎

Remark 1. *Note that the existence conditions are the same but semi-global stabilization allows for a linear controller at the expense of a compact but arbitrarily large domain of attraction.*

3.2. Single layer nonlinear sandwich systems with input saturation

In this subsection, we present two theorems that give necessary and sufficient conditions for solving Problems 1 and 2 for a single layer sandwich system with input saturation.

Theorem 2. *Consider the interconnection of the two systems given by* (3) *and* (4). *Both the semi-global and the global stabilization problems, as formulated in Problems 1 and 2, are solvable if and only if,*

1. *All the eigenvalues of A are in the closed unit disc.*
2. *All the eigenvalues of M are in the closed unit disc.*
3. *The linearized cascaded system is stabilizable, i.e.* $(\mathscr{A},\mathscr{B})$ *is stabilizable where* $\mathscr{A}$ *and* $\mathscr{B}$ *are given by* (11).

Proof : Necessity of the conditions follows along the same lines as in the proof of Theorem 1. Also in this case, sufficiency is established by an explicit construction of a suitable controller in the next section. ■

3.3. Multi-layer nonlinear sandwich systems

In this subsection, we establish the necessary and sufficient conditions for solving Problems 1 and 2 for a multi-layer sandwich system.

Theorem 3. *Consider the interconnection of L_i, $i = 1,\ldots,n$ as given by* (5)*. The semi-global and global stabilization problems defined in 1 and 2 are solvable if and only if*

1. *$(\mathscr{A}_0, \mathscr{B}_0)$ is stabilizable, where*
$$\mathscr{A}_0 = \begin{pmatrix} A_1 & 0 & \ldots & 0 \\ B_2C_1 & A_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & B_nC_{n-1} & A_n \end{pmatrix}, \quad \mathscr{B}_0 = \begin{pmatrix} B_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{12}$$
2. *All A_i have their eigenvalues inside the closed unit disc.*

Proof : The necessity of Conditions 1 and 2 can be proved following the same line as previous theorems. Sufficiency is proved in Section 6 by explicit construction of semi-globally stabilizing controllers. ■

Remark 2. *Note that for all three types of sandwich systems, the solvability conditions for semi-global and global stabilization are the same. The intrinsic difference is that global stabilization, unlike the semi-global stabilization, in general requires a nonlinear state feedback law.*

4. Generalized low-gain design for single layer sandwich systems

4.1. Semi-global stabilization

In this section, we explicitly present a generalized low-gain design for solving Problem 1, concerning semi-global stabilization of the origin of the single-layer sandwich system described by (1), (2). We start by applying a preliminary state feedback $u = Fx + v$ where F is such that $A+BF$ is asymptotically stable. Consider the resulting L_1 system:

$$\begin{aligned} x(k+1) &= (A+BF)x(k) + Bv(k) \\ z(k) &= Cx(k). \end{aligned} \tag{13}$$

We have

$$z(k) = C(A+BF)^k x(0) + \sum_{i=0}^{k-1} C(A+BF)^{k-i-1} Bv(i) \tag{14}$$

$$= C(A+BF)^k x(0) + z_0(k). \tag{15}$$

Define

$$\delta_1 = \frac{1}{2\sum_{k=0}^{\infty} \|C(A+BF)^k B\|}. \tag{16}$$

Since $A+BF$ is asymptotically stable, the above summation is well defined. We know that if

$$\|v(k)\| < \delta_1 \quad \forall k > 0, \tag{17}$$

then $\|z_0(k)\| < \frac{1}{2}$. Next we consider the system

$$\bar{x}(k+1) = \tilde{\mathscr{A}}\bar{x}(k) + \mathscr{B}v(k) \tag{18}$$

where

$$\bar{x}(k) = \begin{pmatrix} x(k) \\ \omega(k) \end{pmatrix}, \quad \tilde{\mathscr{A}} = \begin{pmatrix} A+BF & 0 \\ NC & M \end{pmatrix}, \quad \mathscr{B} = \begin{pmatrix} B \\ 0 \end{pmatrix}. \tag{19}$$

Note that Conditions 1 and 2 of Theorem 1 and asymptotic stability of $A+BF$ together imply that $(\tilde{\mathscr{A}}, \mathscr{B})$ is stabilizable and $\tilde{\mathscr{A}}$ has all its eigenvalues in the closed unit disc.

Our next objective is, for any a priori given compact set $\mathscr{W}$, to find a stabilizing controller for the system (18) such that $\mathscr{W}$ is contained in its domain of attraction and $\|v(k)\| < \delta_1$ for all $k > 0$.

We note that there exists a unique $P_\varepsilon > 0$ satisfying

$$P_\varepsilon = \tilde{\mathscr{A}}' P_\varepsilon \tilde{\mathscr{A}} + \varepsilon I - \tilde{\mathscr{A}}' P_\varepsilon \mathscr{B}(\mathscr{B}' P_\varepsilon \mathscr{B} + I)^{-1} \mathscr{B}' P_\varepsilon \tilde{\mathscr{A}}. \tag{20}$$

The following lemma is already obtained in [17].

Lemma 2. *Consider the system* (18) *with constraint* $\|v(k)\| < \delta_1$, *and assume that* $(\tilde{\mathscr{A}}, \mathscr{B})$ *is stabilizable and* $\tilde{\mathscr{A}}$ *has all its eigenvalues in closed unit disc. For any a priori given compact set* $\bar{\mathscr{W}} \in \mathbb{R}^{n_1+n_2}$, *there exists an* ε^* *such that for any* $0 < \varepsilon < \varepsilon^*$, *the feedback:*

$$v = -(\mathscr{B}' P_\varepsilon \mathscr{B} + I)^{-1} \mathscr{B}' P_\varepsilon \tilde{\mathscr{A}} \bar{x} \tag{21}$$

achieves asymptotic stability of the equilibrium $\bar{x} = 0$ *with* $\bar{\mathscr{W}}$ *contained in its domain of attraction. Moreover, for any initial condition in* $\bar{\mathscr{W}}$, *the constraint does not get violated for any* $k > 0$.

We can now use Lemma 2 to prove that a particular family of control laws achieves semi-global stability of the single-layer nonlinear sandwich system.

Theorem 4. *Consider the interconnection of the two systems given by* (1) *and* (2) *satisfying Conditions 1 and 2 of Theorem 1. Let* F *be an arbitrary matrix such that* $A+BF$ *is asymptotically stable while* $P_\varepsilon > 0$ *is the solution of* (20). *We define what is known as a low-gain state feedback by*

$$u = Fx - (\mathscr{B}' P_\varepsilon \mathscr{B} + I)^{-1} \mathscr{B}' P_\varepsilon \tilde{\mathscr{A}} \bar{x} = F_{1,\varepsilon} x + F_{2,\varepsilon} \omega. \tag{22}$$

For any compact set of initial conditions $\mathscr{W} \in \mathbb{R}^{n_1+n_2}$ *there exists an* $\varepsilon^* > 0$ *such that for all* ε *with* $0 < \varepsilon < \varepsilon^*$ *the controller* (22) *asymptotically stabilizes the equilibrium* $(0,0)$ *with a domain of attraction containing* $\mathscr{W}$.

Proof : Condition 2 of Theorem 1 immediately implies the existence and uniqueness of $P_\varepsilon > 0$ satisfying (20). Moreover, Condition 1 immediately implies $P_\varepsilon \to 0$ as $\varepsilon \to 0$. This immediately implies that $F_{1,\varepsilon} \to F$ and $F_{2,\varepsilon} \to 0$ as $\varepsilon \to 0$. Note that the initial conditions are in some compact set $\mathscr{W}$ and hence there exist compact sets $\mathscr{X}$ and Ω such that $x(0) \in \mathscr{X}$ and $\omega(0) \in \Omega$. Define a family of sets $\mathscr{V}(c_1) = \{\bar{x} \in \mathbb{R}^{n_1+n_2} \mid \bar{x}' P_\varepsilon \bar{x} \leq c_1\}$.

Note that if we apply $u = Fx$, there exists a $K > 0$ such that for any $x(0) \in \mathscr{X}$ we have

$$\|C(A+BF)^k x(0)\| < \tfrac{1}{2}$$

for all $k > K$ and there exists a compact set $\bar{\mathscr{X}}$ such that $x(k) \in \bar{\mathscr{X}}$ for all $0 \leq k \leq K$. This immediately follows from the asymptotic stability of $A+BF$.

Since $\omega(0) \in \Omega$ which is a compact set and $\sigma(z(k))$ is bounded we find that, independent of ε, there exists a compact set $\bar{\Omega}$ such that $\omega(k) \in \bar{\Omega}$ for all $0 \leq k \leq K$.

Next, there exists an $\varepsilon^\# > 0$ such that for $u(k) = F_{1,\varepsilon} x(k) + F_{2,\varepsilon}\omega(k)$ and $\varepsilon < \varepsilon^\#$ we have $x(k) \in 2\bar{\mathscr{X}}$ for all $0 \leq k \leq K$. This follows from the fact that $F_{1,\varepsilon} \to F$ and $F_{2,\varepsilon} \to 0$ while $\omega(k)$ is bounded in $\bar{\Omega}$. Let c_1 be such that

$$c_1 = \sup_{\substack{\varepsilon \in (0,1] \\ \bar{x} \in 2\bar{\mathscr{X}} \times \bar{\Omega}}} \bar{x}' P_\varepsilon \bar{x}.$$

Define a family level sets $\mathscr{V}(c_1) = \{\bar{x} \in \mathbb{R}^{n_1+n_2} \mid \bar{x}' P_\varepsilon \bar{x} \leq c_1\}$.

From Lemma 2, we also note that there exists an $\varepsilon^* < \varepsilon^\#$ such that, for $\varepsilon < \varepsilon^*$, the controller

$$v = -(\mathscr{B}' P_\varepsilon \mathscr{B} + I)^{-1} \mathscr{B}' P_\varepsilon \tilde{\mathscr{A}} \bar{x}$$

stabilizes the system (18), and satisfies $\|v\| < \delta_1$ for all $k > 0$ given $\bar{x}(K) \in \mathscr{V}(c_1)$. This implies that $z(k)$ generated by (13) satisfies $\|z(k)\| < 1$ for $k > K$. Then the interconnection of (1) and (2) with controller (22) for $k > K$ is equivalent to the interconnection of (18) with controller (21) for $k > K$. The asymptotic stability of the latter system follows from Lemma 2. Hence we have $x(k) \to 0, \quad \omega(k) \to 0$. Since this follows for any $(x(0), \omega(0)) \in \mathscr{W}$, we find that $\mathscr{W}$ is contained in the domain of attraction as required. ■

Remark 3. *For semi-global stabilization, we can enlarge the domain of attraction by choosing a sufficiently small low-gain parameter. However, this incurs a deterioration of closed-loop performance near the origin since a small low-gain parameter results in conservativeness in feedback gain and hence does not allow full utilization of control capacity when the state is close to the origin. In order to rectify this problem, a generalized low-and-high gain feedback design methodology for continuous-time sandwich nonlinear systems is recently introduced in [26]. It was shown that a refined performance can be achieved with the so-called low-and-high-gain feedback controller. Because of some inherent differences between continuous- and discrete-time systems, development of a low-and-high-gain design for discrete-time counter-part remains an open research problem.*

Remark 4. *To implement the semi-globally stabilizing controller, it is necessary to find appropriate low-gain parameters ε. It is difficult to derive tight upper bounds on ε analytically, and thus the parameters are typically found experimentally, by gradually decreasing them until the desired stability is achieved.*

4.2. Global stabilization

In what follows, we show that the family of controllers defined by (22), with ε replaced by a scheduled low-gain parameter $\varepsilon(\bar{x})$, solves Problem 2. We consider the scheduling introduced in Section 2 as given by:

$$\varepsilon(\bar{x}) = \max\{\, r \in (0,1] \mid (\bar{x}' P_r \bar{x}) \|\mathscr{B}' P_r \mathscr{B}\| \leq \tfrac{\delta_1^2}{M_p} \,\} \tag{23}$$

where P_r is the unique positive definite solution of algebraic Riccati equation (20) with $\varepsilon = r$, δ_1 is defined by (16),

$$M_p = \sigma_{\max}(P_1^{\frac{1}{2}} \mathscr{B}\mathscr{B}' P_1^{\frac{1}{2}}) + 1$$

and P_1 is the solution of (20) with $\varepsilon = 1$. It has been shown in Section 2 that this scheduling guarantees that

$$\|(\mathscr{B}' P_{\varepsilon(x)} \mathscr{B} + I)^{-1} \mathscr{B}' P_{\varepsilon(x)} \tilde{\mathscr{A}} x\| \leq \delta_1.$$

To prove Theorem 1, we need the following lemma from [19], which defines a control law that stabilizes the linear system (18).

Lemma 3. *Consider the system* (18) *and assume that* $(\tilde{\mathscr{A}}, \mathscr{B})$ *as given by* (19) *is stabilizable, and that the eigenvalues of* $\tilde{\mathscr{A}}$ *are within the closed unit disc. The control law*

$$v = -(\mathscr{B}' P_{\varepsilon(\bar{x})} \mathscr{B} + I)^{-1} \mathscr{B}' P_{\varepsilon(\bar{x})} \tilde{\mathscr{A}} \bar{x} \tag{24}$$

achieves global stability of the equilibrium $\bar{x} = 0$.

We can now use Lemma 3 to prove that a particular family of control laws achieves global stability of the single-layer nonlinear sandwich system.

Theorem 5. *Consider the systems given by* (1) *and* (2)*, satisfying Conditions 1 and 2 of Theorem 1. Choose an arbitrary matrix* F *such that* $A + BF$ *is asymptotically stable. Let* $P_{\varepsilon(\bar{x})}$ *be the unique positive definite solution of ARE* (20)*, with* ε *replaced by the scheduled low-gain parameter* $\varepsilon(\bar{x})$ *defined by* (23)*. Then, the control law*

$$u = Fx - (\mathscr{B}' P_{\varepsilon(\bar{x})} \mathscr{B} + I)^{-1} \mathscr{B}' P_{\varepsilon(\bar{x})} \tilde{\mathscr{A}} \bar{x} \tag{25}$$

achieves global asymptotic stability of the origin where $\tilde{\mathscr{A}}$ *and* $\mathscr{B}$ *are given by* (19).

Proof :

If we consider the interconnection of (1) and (2), then we note that close to the origin the saturation does not get activated. Moreover, close to the origin the feedback (25) is given by:

$$u = Fx - (\mathscr{B}' P_1 \mathscr{B} + I)^{-1} \mathscr{B}' P_1 \tilde{\mathscr{A}} \bar{x}$$

where P_1 is the solution of (20) with $\varepsilon = 1$. This immediately yields that the interconnection of (1), (2) and (25) is locally asymptotically stable. It remains to show that we have global asymptotic stability.

Consider an arbitrary initial condition $x(0)$ and $\omega(0)$. Then there exists a $K > 0$ such that

$$\|C(A+BF)^k x(0)\| < \tfrac{1}{2}$$

for $k > K$. Moreover, by construction

$$v = -(\mathscr{B}' P_{\varepsilon(\bar{x})} \mathscr{B} + I)^{-1} \mathscr{B}' P_{\varepsilon(\bar{x})} \tilde{\mathscr{A}} \bar{x}$$

yields $\|v(k)\| \leq \delta_1$ for all $k > 0$. However, this implies that $z(k)$ generated by (13) satisfies $\|z(k)\| < 1$ for all $k > K$. But this yields that the interconnection of (1) and (2) with controller (25) behaves for

$k > K$ like the interconnection of (18) with controller (24). From Lemma 3, global asymptotic stability of the latter system then implies that $\bar{x}(k) \to 0$ as $k \to \infty$. Since this property holds for any initial condition and we have local asymptotic stability we can conclude that the controller yields global asymptotic stability. This completes the proof. ■

5. Generalized scheduled low-gain design for single layer sandwich systems with input saturation

5.1. Semi-global stabilization

We now present a generalized low-gain design for solving Problem 1 concerning the semi-global stabilization of the origin of the single-layer sandwich system subject to input saturation described by (3), (4).

Let $P_{1,\varepsilon_1} = P'_{1,\varepsilon_1} > 0$ be the unique positive-definite solution of the algebraic Riccati equation

$$P_{1,\varepsilon_1} = A'P_{1,\varepsilon_1}A + \varepsilon_1 I - A'P_{1,\varepsilon_1}B(B'P_{1,\varepsilon_1}B + I)^{-1}B'P_{1,\varepsilon_1}A, \tag{26}$$

and define

$$F_{1,\varepsilon_1} = -(B'P_{1,\varepsilon_1}B + I)^{-1}B'P_{1,\varepsilon_1}A. \tag{27}$$

Next, let $P_{2,\varepsilon_2} = P'_{2,\varepsilon_2} > 0$ be the unique positive-definite solution of the algebraic Riccati equation

$$P_{2,\varepsilon_2} = \tilde{\mathscr{A}}'P_{2,\varepsilon_2}\tilde{\mathscr{A}} + \varepsilon_2 I - \tilde{\mathscr{A}}'P_{2,\varepsilon_2}\mathscr{B}(\mathscr{B}'P_{2,\varepsilon_2}\mathscr{B} + I)^{-1}\mathscr{B}'P_{2,\varepsilon_2}\tilde{\mathscr{A}}, \tag{28}$$

and define

$$F_{2,\varepsilon_2} = -(\mathscr{B}'P_{2,\varepsilon_2}\mathscr{B} + I)^{-1}\mathscr{B}'P_{2,\varepsilon_2}\tilde{\mathscr{A}}, \tag{29}$$

where $\tilde{\mathscr{A}}$ and $\mathscr{B}$ are given by

$$\tilde{\mathscr{A}} = \begin{pmatrix} A + BF_{1,\varepsilon_1} & 0 \\ NC & M \end{pmatrix}, \quad \mathscr{B} = \begin{pmatrix} B \\ 0 \end{pmatrix}.$$

We define the following family of control laws:

$$u = F_{1,\varepsilon_1}x + F_{2,\varepsilon_2}\bar{x}. \tag{30}$$

The family of control laws is parameterized by the parameters $\varepsilon_1, \varepsilon_2 > 0$, and we show in the next theorem that semi-global stabilization is achieved for suitably chosen values of these parameters.

Theorem 6. *Consider the systems given by* (3) *and* (4)*, satisfying Conditions 1, 2, and 3 of Theorem 2. For any compact set of initial conditions* $\mathscr{W} \in \mathbb{R}^{n_1+n_2}$*, there exists an* $\varepsilon_1^* > 0$ *such that for any* ε_1 *with* $0 < \varepsilon_1 < \varepsilon_1^*$*, there exists an* $\varepsilon_2^*(\varepsilon_1)$ *such that for all* $0 < \varepsilon_2 < \varepsilon_2^*(\varepsilon_1)$*, the controller defined by* (30) *asymptotically stabilizes the origin with a domain of attraction containing* $\mathscr{W}$*.*

Proof : By Conditions 1 and 2 of Theorem 2, we know that the eigenvalues of A and M are in the closed unit disc. This implies that $\lim_{\varepsilon_1 \to 0} P_{1,\varepsilon_1} = 0$ and $\lim_{\varepsilon_2 \to 0} P_{2,\varepsilon_2} = 0$, and hence we know that

$$\lim_{\varepsilon_1 \to 0} F_{1,\varepsilon_1} = 0, \qquad \lim_{\varepsilon_2 \to 0} F_{2,\varepsilon_2} = 0. \tag{31}$$

Note that the initial conditions belong to some compact set $\mathscr{W}$, and hence there exist compact sets $\mathscr{X} \subset \mathbb{R}^{n_1}$ and $\Omega \subset \mathbb{R}^{n_2}$ such that $x(0) \in \mathscr{X}$ and $\omega(0) \in \Omega$. Define a family of sets $\mathscr{V}_1(c) = \{x \in \mathbb{R}^{n_1} \mid x'P_{1,\varepsilon_1}x \leq c\}$.

If we apply $u = F_{1,\varepsilon_1}x$, it is proved in [24] that there exists an $\varepsilon_1^* > 0$ such that for all $0 < \varepsilon_1 < \varepsilon_1^*$ and for all $x(0) \in \mathscr{X}$,

$$\|F_{1,\varepsilon_1}(A + BF_{1,\varepsilon_1})^k x(0)\| \leq \tfrac{1}{4}. \tag{32}$$

Moreover, there exists a $K > 0$, dependent on ε_1, such that $x(K) \in \mathscr{V}_1(c_1)$ for all $x(0) \in \mathscr{X}$. Here c_1 is such that $x \in \mathscr{V}_1(c_1)$ implies that $\|Cx\| \leq \frac{1}{4}$ and $\|F_{1,\varepsilon_1}x\| \leq \frac{1}{4}$. Since $\omega(0) \in \Omega$, where Ω is a compact set, and $\sigma(z(k))$ is bounded, it follows that there exists a compact set $\bar{\Omega}$, independent of ε_2, such that $\omega(k) \in \bar{\Omega}$ for all $0 \leq k \leq K$. Define a family of sets

$$\mathscr{V}_2(c) = \left\{ \bar{x} \in \mathbb{R}^{n_1+n_2} \;\mid x'P_{1,\varepsilon_1}x + \bar{x}'P_{2,\varepsilon_2}\bar{x} \leq c \right\}.$$

Next, we note that for $u = F_{1,\varepsilon_1}x$, we have $x(K) \in \mathscr{V}_1(c_1)$. From (31) and our earlier conclusion that $\omega(k)$ is bounded for $0 \leq k \leq K$, we see that if we apply $u = F_{1,\varepsilon_1}x + F_{2,\varepsilon_2}\bar{x}$ then there exists an ε_2^*, dependent on ε_1, such that for all $0 < \varepsilon_2 < \varepsilon_2^*$, the following properties hold:

- $x(K) \in 2\mathscr{V}_1(c_1)$.
- If $x \in 2\mathscr{V}_1(c_1)$ and $\omega \in \bar{\Omega}$, then $\bar{x} \in 3\mathscr{V}_2(c_1)$.
- For any $\bar{x}$ such that $\bar{x} \in 3\mathscr{V}_2(c_1)$, we have $\|F_{2,\varepsilon_2}\bar{x}\| < \frac{1}{4}$.

At time $k = K$, we have $\bar{x} \in 3\mathscr{V}_2(c_1)$. This immediately implies that $\|F_{2,\varepsilon_2}\bar{x}\| \leq \frac{1}{4}$. Note that for any $\bar{x} \in 3\mathscr{V}_2(c_1)$

$$x'P_{1,\varepsilon}x \leq x'P_{1,\varepsilon}x + \bar{x}'P_{2,\varepsilon_2}\bar{x} \leq 9c_1,$$

and hence $x \in 3\mathscr{V}_1(c_1)$. But this implies that $\|F_{1,\varepsilon_1}x\| \leq \frac{3}{4}$. Therefore, we have that $\|F_{1,\varepsilon_1}x + F_{2,\varepsilon_2}\bar{x}\| \leq 1$.

Similarly for any $\bar{x} \in 3\mathscr{V}_2(c_1)$, we have $x \in 3\mathscr{V}_1(c_1)$ and this implies that $\|Cx\| \leq \frac{3}{4}$. Therefore, for any $\bar{x} \in 3\mathscr{V}_2(c_1)$, both saturations are inactive.

We know at time K, the closed-loop system is linear and can be written as

$$\bar{x}(k+1) = (\tilde{\mathscr{A}} + \mathscr{B}F_{2,\varepsilon_2})\bar{x}(k). \tag{33}$$

It is straightforward to see that (33) is asymptotically stable and $3\mathscr{V}_2(c_1)$ is invariant. We know that the two saturations will remain inactive for all $k \geq K$. The asymptotic stability of (33) implies $\bar{x}(k) \to 0$ as $k \to \infty$. Since this holds for any $\bar{x}(0) \in \mathscr{W}$, it follows that $\mathscr{W}$ is contained in the domain of attraction. This completes the proof. ■

5.2. *Global stabilization*

We now present a generalized scheduled low-gain design for solving Problem 2 concerning the global stabilization of the single-layer sandwich system subject to input saturation described by (3), (4). As in previous section, this controller is formed by equipping semi-global controller (30) with scheduled parameters.

Let $P_{1,\varepsilon_1} = P'_{1,\varepsilon_1} > 0$ be the unique positive-definite solution of the algebraic Riccati equation (26) and F_{1,ε_1} be defined as (27) with scheduled parameter $\varepsilon_1 = \varepsilon_1(x)$.

Similar to that in the preceding section, a particular choice of scheduling is given by

$$\varepsilon_1(x) = \max\left\{ r \in (0,1] \mid (x'P_{1,r}x)\|B'P_{1,r}B\| \leq \tfrac{1}{4M_2} \right\} \tag{34}$$

where $P_{1,r}$ is the solution of ARE (26) with $\varepsilon_1 = r$, $M_2 = \sigma_{\max}(P_{1,1}^{\frac{1}{2}} BB' P_{1,1}^{\frac{1}{2}}) + 1$ and $P_{1,1}$ is the solution of ARE (26) with $\varepsilon_1 = 1$. It has been shown that above scheduling guarantees that

$$\left\|(B'P_{1,\varepsilon_1(x)}B+I)^{-1}B'P_{1,\varepsilon_1(x)}x\right\| \leq \tfrac{1}{2}.$$

Let $\ell > 0$ be such that

$$(\lambda_{\max}(P_{1,1}) + \tfrac{1}{2})\ell^2 \leq \tfrac{1}{4M_2\|B'P_{1,1}B\|},$$

and let $P_{2,\varepsilon_2} = P_{2,\varepsilon_2}' > 0$ be the unique positive-definite solution of the algebraic Riccati equation (28) and F_{2,ε_2} be defined by (29) where in both (28) and (29), we take

$$\tilde{\mathscr{A}} = \begin{pmatrix} A+BF_{1,1} & 0 \\ NC & M \end{pmatrix}, \quad \mathscr{B} = \begin{pmatrix} B \\ 0 \end{pmatrix}, \tag{35}$$

and $\varepsilon_2 = \varepsilon_2(\bar{x})$ is a scheduled parameter. Choose

$$\delta_2 = \min\left\{\frac{1}{2}, \frac{\ell^2}{2(3\|B'P_{1,1}B\|+1)}, \frac{1}{2\rho}\right\}, \tag{36}$$

where $\rho = \sum_{k=0}^{\infty}\|C(A+BF_{1,1})^k B\|$. Consider an associated scheduled parameter given by

$$\varepsilon_2(\bar{x}) = \max\{\, r \in (0,1] \mid (\bar{x}'P_{2,r}\bar{x})\|\mathscr{B}'P_{2,r}\mathscr{B}\| \leq \tfrac{\delta_2^2}{M_3}\,\} \tag{37}$$

where $M_3 = \sigma_{\max}(P_{2,1}^{\frac{1}{2}}\mathscr{B}\mathscr{B}'P_{2,1}^{\frac{1}{2}}) + 1$. We have $\|F_{2,\varepsilon}\| \leq \delta_2$. The following theorem shows that a particular control law achieves global stability of the single-layer nonlinear sandwich system subject to input saturation.

Theorem 7. *Consider the two systems given by* (3) *and* (4)*, satisfying Conditions 1, 2 and 3 of Theorem 2. Let $P_{1,\varepsilon_1(x)}$ be the solution of* (26) *with ε_1 replaced by the scheduled low-gain parameter $\varepsilon_1(x)$ defined by* (34)*. Let $P_{2,\varepsilon_2(\bar{x})}$ be the solution of* (28) *with $\tilde{\mathscr{A}}$ and $\mathscr{B}$ given by* (35) *and ε_2 replaced by the scheduled low-gain parameter $\varepsilon_2(\bar{x})$ defined by* (37)*. The control law*

$$u = F_{1,\varepsilon_1(x)}x + \varepsilon_1(x)F_{2,\varepsilon_2(\bar{x})}\bar{x} \tag{38}$$

achieves global asymptotic stability of the origin where $F_{1,\varepsilon_1(x)}$ and $F_{2,\varepsilon_2(\bar{x})}$ are respectively defined by (27) *and* (29) *with ε_1 and ε_2 replaced by $\varepsilon_1(x)$ and $\varepsilon_2(\bar{x})$.*

Proof : Note that our scheduled parameter guarantees that $\|u(k)\| \leq 1$ for all $k \geq 0$. The input saturation is always inactive.

Considering the interconnection of (3) and (4), we note that the sandwiched saturation is not activated near the origin. Moreover, near the origin the control law (38) is given by $u = F_{1,1}x + F_{2,1}\bar{x}$. This means that state matrix of the interconnection of (3), (4), and (38) equals $\tilde{\mathscr{A}} + \mathscr{B}F_{2,1}$ which is asymptotically stable by the properties of the algebraic Riccati equation. We have therefore established local asymptotic stability. It remains to show that we have global asymptotic stability.

Define $V = x'P_{\varepsilon_1(x)}x$ and $\mathscr{V}_1 = \{x \in \mathbb{R}^{n_1} \mid \|x\| \leq \ell\}$ and $\mathscr{V}_2 = \{x \in \mathbb{R}^{n_1} \mid V(x) \leq (\lambda_{\max}(P_{1,1}) + 1/2)\ell^2\}$. Since $\|x(k)\| \leq \ell$ implies that $V(x) \leq \lambda_{\max}(P_{\varepsilon_1(x(k))})\|x(k)\|^2 \leq \lambda_{\max}(P_{1,1})\ell^2$, we have that $\mathscr{V}_2 \supset \mathscr{V}_1$. Moreover, from definition of ℓ, we have that $\varepsilon_1(x) = 1$ for $x \in \mathscr{V}_2$. We first want to establish that $V(k)$ is strictly decreasing in time when $x \notin \mathscr{V}_1$.

Assume that this is not the case and we can find $x(k) \notin \mathscr{V}_1$ such that $V(k+1)-V(k) \geq 0$. Denote $\varepsilon_1(x(k))$ and $P_{1,\varepsilon_1(x(k))}$ by $\varepsilon_1(k)$ and $P_1(k)$ respectively. We obtain

$$\begin{aligned}V(k+1)-V(k) \leq -\varepsilon_1(k)x(k)'x(k) - x(k+1)'P_1(k)x(k+1) + x(k+1)'P_1(k+1)x(k+1) - \\ 2x(k)'A'P_1(k)Bv_2(k) - 2v_1(k)'B'P_1(k)Bv_2(k) + v_2(k)'B'P_1(k)Bv_2(k)\end{aligned}$$

where $v_1(k) = F_{1,\varepsilon_1(k)}x(k)$ and $v_2(k) = -\varepsilon_1(k)F_{2,\varepsilon_2(k)}\bar{x}(k)$.

Our scheduling guarantees that $\|v_1(k)\| \leq \frac{1}{2}$ and $\|v_2(k)\| \leq \varepsilon_1(k)\delta_2$ and hence

$$\begin{aligned}&\|x(k)'A'P_1(k)Bv_2(k)\| = \|v_1(k)'(B'P_1(k)B+I)v_2(k)\| \leq \tfrac{1}{2}\varepsilon_1(k)(\left\|B'P_{1,1}B\right\|+1)\delta_2 \\ &\|v_1(k)'B'P_1(k)B'v_2(k)\| \leq \tfrac{1}{2}\varepsilon_1(k)\left\|B'P_{1,1}B\right\|\delta_2 \\ &\|v_2(k)'B'P_1(k)Bv_2(k)\| \leq \varepsilon_1(k)^2\left\|B'P_{1,1}B\right\|\delta_2^2 \leq \varepsilon_1(k)\left\|B'P_{1,1}B\right\|\delta_2.\end{aligned} \tag{39}$$

Therefore

$$\begin{aligned}&V(k+1)-V(k) \\ &\leq -\varepsilon_1(k)x'(k)x(k) + x(k+1)'(P_1(k+1)-P_1(k))x(k+1) + \varepsilon_1(k)(3\left\|B'P_{1,1}B\right\|+1)\delta_2 \\ &\leq -\varepsilon_1(k)x'(k)x(k) + x(k+1)'(P_1(k+1)-P_1(k))x(k+1) + \tfrac{1}{2}\varepsilon_1(k)\ell^2 \\ &\leq -\tfrac{1}{2}\varepsilon_1(k)\|x(k)\|^2 + x(k+1)'(P_1(k+1)-P_1(k))x(k+1),\end{aligned} \tag{40}$$

where we use that $x(k) \notin \mathscr{V}_1$ and hence $\|x(k)\| \geq \ell$. Since $V(k+1)-V(k) \geq 0$, the properties of our scheduling imply that $x(k+1)'(P_1(k+1)-P_1(k))x(k+1) \leq 0$. We get

$$V(k+1)-V(k) \leq -\tfrac{1}{2}\varepsilon_1(k)\|x(k)\|^2 < 0.$$

This yields a contradiction. Hence when $x(k) \notin \mathscr{V}_1$ we have that $V(k)$ is strictly decreasing, and it follows that $x(k)$ enters $\mathscr{V}_1$ within finite time, say K_1. When $x(k) \in \mathscr{V}_1$, we have either $V(k+1)-V(k) \leq 0$ or $x(k+1)'(P_1(k+1)-P_1(k))x(k+1) \leq 0$, and (40) yields that

$$V(k+1)-V(k) \leq \tfrac{1}{2}\varepsilon_1(k)\ell^2 \leq \tfrac{1}{2}\ell^2.$$

This implies that $V(k+1) \leq \lambda_{\max}(P_{1,1})\ell^2 + \frac{1}{2}\ell^2$ and hence $x(k+1) \in \mathscr{V}_2$. We find that if $x(k) \in \mathscr{V}_1$ then $x(k+1) \in \mathscr{V}_2$. On the other hand, if $x(k) \in \mathscr{V}_2 \backslash \mathscr{V}_1$ then $V(k)$ is strictly decreasing and hence $x(k+1) \in \mathscr{V}_2$. Therefore, $x(k)$ will enter $\mathscr{V}_2$ and it cannot escape from $\mathscr{V}_2$. On $\mathscr{V}_2$ we have $\varepsilon_1(k) = 1$. The L_1 system then becomes:

$$\begin{aligned}x(k+1) &= (A+BF_{1,1})x(k) + Bv_2(k) \\ z(k) &= Cx(k),\end{aligned} \tag{41}$$

where $\|v_2(k)\| \leq \delta_2$. We have for any $k > K_1$

$$z(k) = C(A+BF_{1,1})^{k-K_1}x(K_1) + z_0(k)$$

where

$$z_0(k) = \sum_{i=K_1}^{k-1} C(A+BF_{1,1})^{k-i-1}Bv_2(i). \tag{42}$$

Given that $\delta_2 \leq \frac{1}{2\rho}$ as given by (36), we have $\|v(k)\| < \frac{1}{2\rho}$ for all $k > K_1$. But this guarantees that $\|z_0(k)\| < \frac{1}{2}$ for all $k > K_1$, where $z_0(k)$ is defined by (42). Therefore there exists a K_2 such that for $k \geq K_2$

$$\|C(A+BF_{1,1})^{k-K_1}x(K_1)\| \leq \tfrac{1}{2}$$

and hence $\|z(k)\| \leq 1$ for $k \geq K_2$. We can then apply Lemma 3 as in the previous subsection, and we conclude that the system therefore behaves like a stable system after a finite amount of time, and it follows that $x(k) \to 0$ and $\omega(k) \to 0$ as $k \to \infty$. ■

6. Generalized low-gain design for multi-layer sandwich systems

6.1. Semi-global stabilization

Now we construct a linear semi-globally stabilizing controller for the multi-layer sandwich system which solves semi-global stabilization as formulated in Problem 1.

Consider the interconnection of L_i as defined in (5). Let P_i be the positive definite solution of Riccati equation

$$P_{\varepsilon_i} = \mathscr{A}_i' P_{\varepsilon_i} \mathscr{A}_i + \varepsilon_i I - \mathscr{A}_i' P_{\varepsilon_i} \mathscr{B}_i (\mathscr{B}_i' P_{\varepsilon_i} \mathscr{B}_i + I)^{-1} \mathscr{B}_i' P_{\varepsilon_i} \mathscr{A}_i \tag{43}$$

and define

$$F_{\varepsilon_i} = -(\mathscr{B}_i' P_{\varepsilon_i} \mathscr{B}_i + I)^{-1} \mathscr{B}_i' P_{\varepsilon_i} \mathscr{A}_i \tag{44}$$

where

$$\mathscr{A}_1 = A_1, \quad \mathscr{A}_i = \begin{pmatrix} \mathscr{A}_{i-1} + \mathscr{B}_{i-1} F_{\varepsilon_{i-1}} & 0 \\ B_i \mathscr{C}_{i-1} & A_i \end{pmatrix} \quad \text{for } i = 2, \ldots, n$$

and

$$\mathscr{B}_i = \begin{pmatrix} B_1' & 0 & \cdots & 0 \end{pmatrix}', \quad \mathscr{C}_i = \begin{pmatrix} 0 & \ldots & 0 & C_i \end{pmatrix} \tag{45}$$

are of appropriate dimensions. The parameters ε_i, $i = 1, \ldots, n$ are to be determined appropriately shortly. We have the following theorem:

Theorem 8. *Consider interconnection of n systems as given by* (5)*, satisfying Conditions 1, 2 of Theorem 3. Let P_{ε_i} be the solution of Riccati equations in* (43) *with $\varepsilon_i \in (0,1]$, $i = 1, \ldots, n$. For any compact set $\mathscr{W} \subset \mathbb{R}^{\sum_{i=1}^n n_i}$, we can determine ε_i, $i = 1, \ldots, n$ such that the controller*

$$u = \sum_{i=1}^{n} F_{\varepsilon_i} \chi_i \tag{46}$$

renders the origin asymptotically stable with a domain of attraction containing $\mathscr{W}$ where

$$\chi_i = \begin{pmatrix} x_1' & \cdots & x_i' \end{pmatrix}'. \tag{47}$$

Proof : For simplicity of presentation, denote P_{ε_i} and F_{ε_i} by P_i and F_i.

Conditions (1) and (2) of Theorem 3 and the fact that $\mathscr{A}_i+\mathscr{B}_iF_i$ is asymptotically stable imply that

$$\lim_{\varepsilon_i\to 0} P_i=0, \quad \lim_{\varepsilon_i\to 0} F_i=0 \tag{48}$$

Define function $V_i(\chi_i)=\sum_{j=1}^{i}\chi_j'P_j\chi_j$ and set $\mathscr{V}_i(c)=\left\{\chi_i\in\mathbb{R}^{\sum_{j=1}^{i}n_j}\mid V_i(\chi_i)\le c\right\}$.

Since $\mathscr{W}$ is compact, there exist for $i=1,\ldots,n$, compact sets $\mathscr{W}_i$ such that $\chi_n(0)\in\mathscr{W}$ implies that $x_i(0)\in\mathscr{W}_i$. Next we determine ε_i recursively.

Determine ε_1: Let's consider applying a controller $v_1=F_1\chi_1=F_1x_1$.

Note that (48) implies the existence of an ε_1^* such that for any $\varepsilon\in(0,\varepsilon_1^*]$ and $x_1(0)\in\mathscr{W}_1$, we have

$$\|F_1(A_1+B_1F_1)^kx_1(0)\|\le\tfrac{1}{4^{n-1}}$$

for all $k\ge 0$. Let c_1 be such that, $x_1\in\mathscr{V}_1(c_1)$ implies $\|F_1x_1\|\le\frac{1}{4^{n-1}}$ and $\|C_1x_1\|\le\frac{1}{3^{n-1}}$. Since $A_1+B_1F_1$ is asymptotically stable, there exists a K_1 such that for all $x_1\in\mathscr{W}_1$, we have $x_1(K_1)\in\mathscr{V}_1(c_1)$.

Determine ε_2: Since $x_2(0)\in\mathscr{W}_2$ and the input to L_2 is bounded, there exists a $\bar{\mathscr{W}}_2$ such that

$$x_2(k)\in\bar{\mathscr{W}}_2, \quad \text{for } \ k\le K_1.$$

Let ε_1 be fixed. Consider applying the controller $v_2=F_1x_1+F_2\chi_2$. Due to (48), given $x_2\in\mathscr{W}_2$, there exists an $\varepsilon_2^*(\varepsilon_1)$ such that the following properties hold:

1. For any $\varepsilon_2\in(0,\varepsilon_2^*(\varepsilon_1)]$, $x_1(K_1)\in 2\mathscr{V}_1(c_1)$.
2. For any $\varepsilon_2\in(0,\varepsilon_2^*(\varepsilon_1)]$, $x_1\in 2\mathscr{V}_1(c_1)$ and $x_2\in\bar{\mathscr{W}}_2$ imply $\chi_2\in 3\mathscr{V}_2(c_1)$.
3. For any $\varepsilon_2\in(0,\varepsilon_2^*(\varepsilon_1)]$, $\chi_2\in 3\mathscr{V}_2(c_1)$ implies $\|F_2\chi_2\|\le\frac{1}{4^{n-1}}$.

At time $k=K_1$, we know $\chi_2\in 3\mathscr{V}_2(c_1)$. For any $\chi_2\in 3\mathscr{V}_2(c_1)$, $\|F_2\chi_2\|\le\frac{1}{4^{n-1}}$. Also note that $\chi_2\in 3\mathscr{V}_2(c_1)$ implies then $V_1(x_1)\le 9c_1$ and hence $F_1x_1\le\frac{3}{4^{n-1}}$ and $\|C_1x_1\|\le\frac{1}{3^{n-2}}$. We have

$$\|u\|=\|F_1x_1+F_2\chi_2\|\le\tfrac{1}{4^{n-2}}.$$

Therefore two saturations are both inactive in $3\mathscr{V}_2(c_1)$, it is straightforward to see that with controller v_2, $\chi_2(k)\in 3\mathscr{V}_2(c_1)$ for all $k\ge K_1$ and moreover $\chi_2(k)\to 0$ as $k\to\infty$.

Let c_2 be such that $\chi_2'P_2\chi_2\le c_2$ implies $\|C_1x_1\|\le\frac{1}{3^{n-2}}$, $\|\mathscr{C}_2\chi_2\|\le\frac{1}{3^{n-2}}$. There exists a K_2 such that for all $\chi_2(K_1)\in 3\mathscr{V}_2(c_1)$, we have $\chi_2(K_2)\in\mathscr{V}_2(c_2)$. At time K_2, we get

1. $\chi_2(K_2)\in\mathscr{V}_2(c_2)$.
2. $\|C_1x_1(K_2)\|\le\frac{1}{3^{n-2}}$ and $\|\mathscr{C}_2\chi_2(K_2)\|\le\frac{1}{3^{n-2}}$.
3. $\|F_1x_1+F_2\chi_2\|\le\frac{1}{4^{n-2}}$ for all $\chi_2\in\mathscr{V}_2(c_2)$ and $k\le K_2$.

Determine $\varepsilon_3,\ldots,\varepsilon_n$: Consider system L_i, $i\ge 3$. At this moment, ε_j, c_j and K_j for $j\le i-1$ have been determined in previous $i-1$ steps. The resulting controller $v_{i-1}=\sum_{j=1}^{i-1}F_j\chi_j$ yields

1. $\chi_{i-1}(K_{i-1})\in\mathscr{V}_{i-1}(c_{i-1})$.
2. $\|\mathscr{C}_j\chi_j(K_{i-1})\|\le\frac{1}{3^{n-i+1}}$ for all $j\le i-1$.
3. $\|\sum_{j=1}^{i-1}F_j\chi_j\|\le\frac{1}{4^{n-i+1}}$ for all $\chi_{i-1}\in\mathscr{V}_{i-1}(c_{i-1})$.

Since the input to L_i is bounded and $x_i(0) \in \mathscr{W}_i$, we know that there exists a $\bar{\mathscr{W}_i}$ such that $x_i(k) \in \bar{\mathscr{W}_i}$ for all $k \leq K_{i-1}$. Consider the controller $v_i = \sum_{j=1}^{i} F_j \chi_j$. Then (48) implies the existence of an $\varepsilon_i^*(\varepsilon_1, \ldots, \varepsilon_{i-1})$ such that the following properties hold:

1. $\chi_{i-1}(K_{i-1}) \in 2\mathscr{V}_{i-1}(c_{i-1})$.
2. $\chi_{i-1} \in 2\mathscr{V}_{i-1}(c_{i-2})$ and $x_i \in \bar{\mathscr{W}_i}$ imply that $\chi_i \in 3\mathscr{V}_i(c_{i-1})$.
3. $\chi_i \in 3\mathscr{V}_i(c_{i-1})$ implies $F_i\chi_i \leq \frac{1}{4^{n-i+1}}$.

Therefore, we get at $k = K_{i-1}$, $\chi_i(K_{i-1}) \in 3\mathscr{V}_i(c_{i-1})$, i.e $V_i(\chi) \leq 9c_{i-1}$. But this implies that $V_{i-1}(\chi) \leq 9c_{i-1}$. Hence we have $\|\mathscr{C}_j\chi_j\| \leq \frac{1}{3^{n-i}}$ for all $j = 1, \ldots, i-1$ and that $\|F_i\chi_i\| \leq \frac{3}{4^{n-i+1}}$. Moreover

$$\|v_i\| = \|F_j\chi_i\| + \|\sum_{j=1}^{i-1} F_j\chi_j\| \leq \frac{1}{4^{n-i+1}} + \frac{3}{4^{n-i+1}} = \frac{1}{4^{n-i}}.$$

In conclusion, the first i saturations are inactive for any $\chi_i \in 3\mathscr{V}_i(c_{i-1})$. It is easy to see that with controller v_i, $\chi(k) \in 3\mathscr{V}_i(c_{i-1})$ for all $k \geq K_{i-1}$ and moreover $\chi_i(k) \to 0$ as $k \to \infty$.

Let c_i be such that $V_i(\chi_i) \leq c_i$ implies that $\|\mathscr{C}_j\chi_j\| \leq \frac{1}{3^{n-i}}$ for all $j \leq i$. There exists a K_i such that $\chi_i(K_i) \in \mathscr{V}(c_i)$ for all $\chi_i(K_{i-1}) \in 3\mathscr{V}_i(c_{i-1})$. At time K_i, we have

1. $\chi_i(K_i) \in \mathscr{V}_i(c_i)$.
2. $\|\mathscr{C}_j\chi_j(K_{i-1})\| \leq \frac{1}{3^{n-i}}$ for all $j \leq i$.
3. $\|\sum_{j=1}^{i} F_j\chi_j\| \leq \frac{1}{4^{n-i}}$ for all $\chi_i \in \mathscr{V}_i(c_i)$.

Repeating this procedure, we can determine $\varepsilon_1, \ldots, \varepsilon_n$, c_n, K_n and a controller $u(\chi_n) = v_n(\chi_n) = \sum_{i=1}^{n} F_i\chi_i$ such that for $k \geq K_n$ we have:

1. $\chi_n(K_n) \in \mathscr{V}_n(c_n)$.
2. $\|\mathscr{C}_j\chi_j(K_n)\| \leq 1$ for all $j \leq n$.
3. $\|\sum_{j=1}^{n} F_j\chi_j\| \leq 1$ for all $\chi_{i-1} \in 3\mathscr{V}_{i-1}(c_0)$.

Then the interconnection of n systems is equivalent to

$$\dot{\chi} = (\mathscr{A}_n + \mathscr{B}_n F_n)\chi_n.$$

The stability of this system implies that $\chi_n(k) \to 0$ as $k \to \infty$. This completes the proof. ∎

6.2. Global stabilization

In this section we construct global stabilizing controller for multi-layer systems to prove sufficiency of Conditions 1 and 2 in Theorem 3. This controller is formed by assembling semi-global stabilizing controller (46) with scheduled parameters.

Let $P_{\varepsilon_i(\chi_i)}$ be the positive definite solution of Riccati equation (43) and $F_{\varepsilon_i(\chi_i)}$ be defined by (44) where $\varepsilon_i = \varepsilon_i(\chi_i)$ is a scheduled parameter, $\mathscr{B}$ is given by (45) and

$$\mathscr{A}_1 = A_1, \quad \mathscr{A}_i = \begin{pmatrix} \mathscr{A}_{i-1} + \mathscr{B}_{i-1}F_{i-1,1} & 0 \\ B_i\mathscr{C}_{i-1} & A_i \end{pmatrix}, \quad i = 2, \ldots, n \tag{49}$$

where $F_{i,1} = -(\mathscr{B}_i' P_{\varepsilon_i}\mathscr{B}_i + I)^{-1}\mathscr{B}_i' P_{\varepsilon_i}\mathscr{A}_i$ with $\varepsilon_i = 1$.

We need n scheduled parameters which satisfy similar properties as given in Section 2. Choose

$$\delta_1 = \tfrac{1}{n}, \quad \delta_i = \min\left\{\frac{1}{n},\, \delta_{i-1},\, \frac{\ell_{i-1}^2}{2(n-i+1)^2(\frac{n+2}{n}\left\|\mathscr{B}_{i-1}'P_{i-1,1}\mathscr{B}_{i-1}\right\|+\frac{2}{n})}, \frac{1}{2(n-i+1)\rho_{i-1}}\right\} \tag{50}$$

for $i = 2,\dots,n$ where ℓ_i is such that

$$(\lambda_{\max}(P_{i,1})+\tfrac{1}{2})\ell_i^2 \le \tfrac{\delta_i^2}{M_i\|B'P_{i,1}B\|}$$

and

$$\rho_i = \sum_{k=0}^{\infty}\|\mathscr{C}_i(\mathscr{A}_i+\mathscr{B}_iF_{i,1})^k\mathscr{B}_i\|\}.$$

Consider the following scheduled parameters

$$\varepsilon_i(\chi_i) = \max\{\, r\in(0,1] \mid (\chi_i'P_r\chi_i)\|\mathscr{B}_i'P_r\mathscr{B}_i\| \le \tfrac{\delta_i^2}{M_i}\,\} \tag{51}$$

where χ_i is given by (47), P_r is the solution of (43) with $\varepsilon_i = r$, $M_i = \sigma_{\max}(P_{i,1}^{\frac{1}{2}}\mathscr{B}_i\mathscr{B}_i'P_{i,1}^{\frac{1}{2}})+1$ and $P_{i,1}$ is the solution of (43) with $\varepsilon_i = 1$. Consider the controller

$$u_1 = \sum_{i=1}^{n}(\prod_{j=0}^{i-1}\varepsilon_j(\chi_j))F_{\varepsilon_i(\chi_i)}\chi_i \tag{52}$$

with $\varepsilon_0 = 1$. It has been shown that our scheduling with δ_i defined in (50) guarantees that $\|F_{\varepsilon_i(\chi_i)}\chi_i\| \le \frac{1}{n}$ and hence $\|u_1\| \le 1$. This implies that the input saturation to the first system never gets activated. The following theorem shows that the controller (52) with tuning parameters defined by (51) achieves global asymptotic stability of the origin for multi-layer nonlinear sandwich system.

Theorem 9. *Consider interconnection of system L_i given in* (5)*, satisfying Conditions 1, 2 of Theorem 3. The control* (52) *achieves global asymptotic stability of the origin.*

Proof : For the simplicity of presentation, we denote $\varepsilon_i(\chi_i(k))$, $P_{\varepsilon_i(\chi_i(k))}$ and $F_{\varepsilon_i(\chi_i(k))}$ by $\varepsilon_i(k)$, $P_i(k)$ and $F_i(k)$ respectively. But we emphasize that they always depend on χ_i.

When the state is sufficiently close to the origin, all saturation elements are inactive and $\varepsilon_i(\chi_i) = 1$ for all $i = 1,\dots,n$. The state matrix of the closed-loop system is given by $\mathscr{A}_n + \mathscr{B}_nF_{n,1}$. From the property of ARE, we know that the above matrix is asymptotically stable. Then local stability follows.

We shall prove global attractivity using induction. We have argued that for all $k \ge 0$, the input saturation on L_1 remains inactive and by construction $\varepsilon_0 = 1$. Suppose there exists a K_i with $1 \le i \le n-1$ such that $\varepsilon_j = 1$ for $j \le i-1$ and the first i saturations are inactive for all $k \ge K_i$. We shall show that there exists a K_{i+1} such that $\varepsilon_i = 1$ and saturation on L_{i+1} will be inactive for all $k \ge K_{i+1}$. By assumption, for $k \ge K_i$, the interconnection of first i systems is equivalent to the following linear system

$$\dot{\chi}_i = \mathscr{A}_i\chi_i + \mathscr{B}_iv_1 \tag{53}$$

where $\mathscr{A}_i$ is given by (49) and v_1 is given by

$$v_1 = v_{1,1} + v_{1,2} = F_i\chi_i + \sum_{j=i+1}^{n}(\prod_{t=i}^{j-1}\varepsilon_t)F_j\chi_j.$$

Define $V_i(k) = \chi_i' P_i \chi_i$ and the family of sets $\mathscr{V}_{i,1} = \{ \chi_i \in \mathbb{R}^{\sum_{j=1}^{i} n_j} \mid \|\chi_i\| \leq \ell_i \}$ and $\mathscr{V}_{i,2} = \{ \chi_i \in \mathbb{R}^{\sum_{j=1}^{i} n_j} \mid V_i \leq (\lambda_{\max}(P_{i,1}) + 1/2)\ell_i^2 \}$. Since $x(k) \in \mathscr{V}_{i,1}$ implies

$$V_i(k) \leq \lambda_{\max}(P_i(k))\|\chi_i(k)\|^2 \leq \lambda_{\max}(P_{i,1})\ell_i^2,$$

we find that $\mathscr{V}_{i,1} \subset \mathscr{V}_{i,2}$. Moreover, the definition of ℓ_i implies that $\varepsilon_i(k) = 1$ for $\chi_i(k) \in \mathscr{V}_{i,2}$.

Evaluating $V_i(k+1) - V_i(k)$ along the trajectories yields:

$$\begin{aligned} V_i(k+1) - V_i(k) \leq{}& -\varepsilon_i(k)\chi_i(k)'\chi_i(k) - \chi_i(k+1)'P_i(k)\chi_i(k+1) + \chi_i(k+1)'P_i(k+1)\chi_i(k+1) - \\ & 2\chi_i(k)'\mathscr{A}_i'P_i(k)\mathscr{B}_i v_{1,2}(k) - 2v_{1,1}(k)'\mathscr{B}_i'P_i(k)\mathscr{B}_i v_{1,2}(k) + v_{1,2}(k)'\mathscr{B}_i'P_i(k)\mathscr{B}_i v_{1,2}(k) \end{aligned}$$

where

$$v_{1,1}(k) = F_i(k)\chi_i(k), \qquad v_{1,2}(k) = \sum_{j=i+1}^{n} \Big(\prod_{t=i}^{j-1} \varepsilon_t(k)\Big) F_j(k)\chi_j(k).$$

Our scheduling guarantees that $\|v_{1,1}(k)\| \leq \frac{1}{n}$ and $\|v_{1,2}(k)\| \leq \varepsilon_i(k)(n-i)\delta_{i+1}$ and hence

$$\begin{aligned} &\|\chi_i(k)'\mathscr{A}_i'P_i(k)\mathscr{B}_i v_{1,2}(k)\| = \|v_{1,1}(k)'(\mathscr{B}_i'P_i(k)\mathscr{B}_i + I)v_{1,2}(k)\| \leq \varepsilon_i(k)\tfrac{(n-i)^2}{n}(\left\|\mathscr{B}_i'P_i(k)\mathscr{B}_i\right\| + 1)\delta_{i+1} \\ &\|v_{1,1}(k)'\mathscr{B}_i'P_i(k)\mathscr{B}_i v_{1,2}(k)\| \leq \varepsilon_i(k)\tfrac{(n-i)^2}{n}\left\|\mathscr{B}_i'P_i(k)\mathscr{B}_i\right\|\delta_{i+1} \\ &\|v_{1,2}(k)'\mathscr{B}_i'P_i(k)\mathscr{B}_i v_{1,2}(k)\| \leq \varepsilon_i(k)(n-i)^2\left\|\mathscr{B}_i'P_i(k)\mathscr{B}_i\right\|\delta_{i+1}. \end{aligned}$$

With the above inequalities, we have

$$\begin{aligned} & V_i(k+1) - V_i(k) \\ \leq{}& -\varepsilon_i(k)\|\chi_i(k)\|^2 + \chi_i(k+1)'(P_i(k+1) - P_i(k))\chi_i(k+1) + \varepsilon_i(k)(n-i)^2\left(\tfrac{n+4}{n}\left\|\mathscr{B}_i'P_i(k)\mathscr{B}_i\right\| + \tfrac{2}{n}\right)\delta_{i+1} \\ \leq{}& -\varepsilon_i(k)\|\chi_i(k)\|^2 + \chi_i(k+1)'(P_i(k+1) - P_i(k))\chi_i(k+1) + \tfrac{1}{2}\varepsilon_i(k)\ell_i^2. \end{aligned}$$

Using the same argument as in the proof of Theorem 7, we can show that if $\chi_i(k) \notin \mathscr{V}_{i,1}$ then $V_i(k)$ is strictly decreasing and hence χ_i will enter $\mathscr{V}_{i,1}$ within finite time. On the other hand, if $\chi_i(k) \in \mathscr{V}_{i,1}$ then $\chi_i(k+1) \in \mathscr{V}_{i,2}$. Since $\mathscr{V}_{i,1} \subset \mathscr{V}_{i,2}$, we conclude that χ_i will enter $\mathscr{V}_{i,2}$ within finite time, say $K_{i,1}$, and can not escape from it. On $\mathscr{V}_{i,2}$ we have $\varepsilon_i(k) = 1$.

Consider $z_i(k) = C_i x_i(k) = \mathscr{C}_i\chi_i(k)$ for $k \geq K_{i,1}$. Since $\varepsilon_i(k) = 1$, we have

$$z_i(k) = \mathscr{C}_i(\mathscr{A}_i + \mathscr{B}_i F_{i,1})^{k-K_{i,1}}\chi_i(K_{i,1}) + z_{i,0}(k)$$

where

$$z_{i,0}(k) = \sum_{j=K_{i,1}}^{k-1} \mathscr{C}_i(\mathscr{A}_i + \mathscr{B}_i F_{i,1})^{k-j-1}\mathscr{B}_i v_{1,2}(j).$$

Our scheduling guarantees that

$$v_{1,2} \leq (n-i)\delta_{i+1} \leq \tfrac{1}{2\rho_i} = \frac{1}{2\sum_{k=0}^{\infty}\|\mathscr{C}_i(\mathscr{A}_i - \mathscr{B}_i F_{i,1})^k\mathscr{B}_i\|}.$$

This implies that $\|z_{i,0}(k)\| \leq \frac{1}{2}$ for all $k \geq K_{i,1}$. Since $\mathscr{A}_i + \mathscr{B}_i F_{i,1}$ is asymptotically stable, there exists a $K_{i+1} > K_{i,1}$ such that for all $k \geq K_i$, we have $\|z_i(k)\| \leq 1$.

Therefore the input saturation on L_{i+1} will be inactive and $\varepsilon_i = 1$ for all $k \geq K_{i+1}$. By induction, there exists a K_n such that all the saturations are inactive for $k \geq K_n$ and $\varepsilon_i = 1$ for all $i = 0, \ldots, n-1$.

Then the interconnection of n systems (5) and controller (52) is equivalent to the interconnection of linear system

$$\chi_n(k+1) = \mathscr{A}_n \chi_n(k) + \mathscr{B}_n v_1$$

with controller $v_1 = F_{\varepsilon_n(\chi_n)} \chi_n = -(\mathscr{B}_n' P_{\varepsilon_n(\chi_n)} \mathscr{B}_n + I)^{-1} \mathscr{B}_n' P_{\varepsilon_n(\chi_n)} \mathscr{A}_n \chi_n$. It follows from Lemma 3 that the closed-loop system is globally asymptotically stable, i.e. $\chi_n(k) \to 0$ as $k \to \infty$. This shows global attractivity of the origin and completes the proof. ■

7. Example

7.1. Single layer sandwich system

Consider the following two systems:

$$L_1 : \begin{cases} x(k+1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x(k) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(k), \\ \quad z(k) \quad = \begin{pmatrix} 1 & 0 \end{pmatrix} x(k), \end{cases} \tag{54}$$

and

$$L_2 : \quad \omega(k+1) = \begin{pmatrix} 0.8 & 0.6 \\ -0.6 & 0.8 \end{pmatrix} \omega(k) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sigma(z(k)), \tag{55}$$

and $\mathscr{W} = [-2,2] \times [-2,2] \times [-2,2] \times [-2,2]$. We shall design controllers for both semi-global and global stabilization of the origin of (54) and (55). The initial condition for simulations is $x(0) = (-2,2)'$ and $\omega(0) = (2,-2)'$.

7.1.1. Semi-global stabilization

- Choose $F = \begin{pmatrix} -0.7321 & 0 \end{pmatrix}$.
- From (16), we calculate $\delta_1 = 0.366$.
- Determine ε according to $\mathscr{W}$ and δ_1. We choose $\varepsilon = 3 \times 10^{-3}$.
- The feedback controller is given by

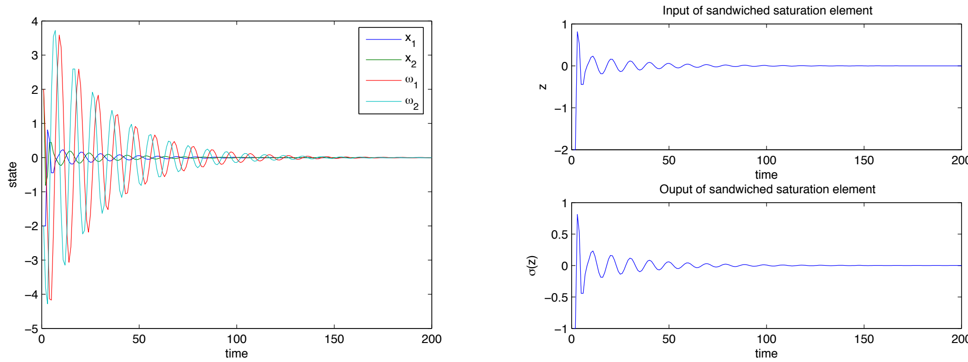
$$u = \begin{pmatrix} -0.7145 & -0.055 & -0.0740 & -0.0087 \end{pmatrix} \bar{x}.$$

The simulation data is shown in Figure 5.

7.1.2. Global stabilization

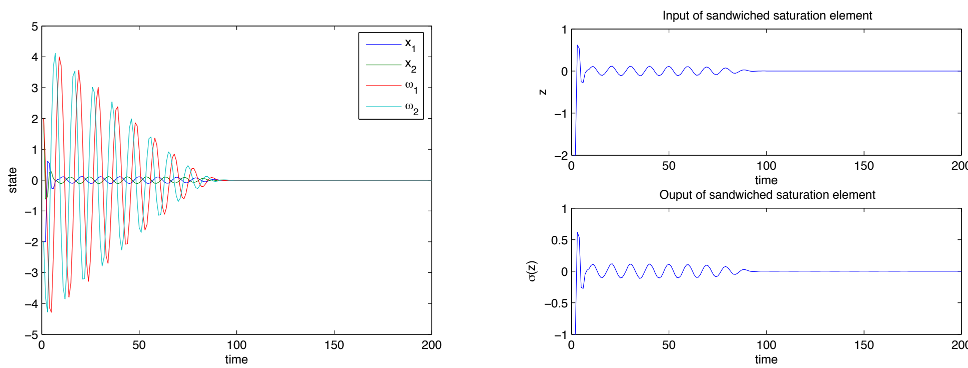
- Choose $F = \begin{pmatrix} -0.7321 & 0 \end{pmatrix}$.
- From (16), we compute $\delta_1 = 0.366$.
- The global stabilizing controller is formed by semi-global controller together with scheduled parameter.

The simulation data is shown in Figure 6 and 7.



State evolution I/O of sandwiched saturation

Figure 5. Semi-global stabilization of single layer sandwich system



State evolution I/O of saturation

Figure 6. Global stabilization of single layer sandwich system

7.2. *Single layer sandwich system with input saturation*

Consider the systems as given by (56) and (57)

$$L_1 : \begin{cases} x(k+1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x(k) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sigma(u(k)), \\ z(k) \quad = \begin{pmatrix} 1 & 0 \end{pmatrix} x(k), \end{cases} \tag{56}$$

and

$$L_2 : \quad \omega(k+1) = \begin{pmatrix} 0.8 & 0.6 \\ -0.6 & 0.8 \end{pmatrix} \omega(k) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sigma(z(k)), \tag{57}$$

and $\mathscr{W} = [-2,2] \times [-2,2] \times [-2,2] \times [-2,2]$. The initial condition for simulation is $x(0) = \begin{pmatrix} -2 & 2 \end{pmatrix}'$ and $\omega(0) = \begin{pmatrix} 2 & -2 \end{pmatrix}'$.

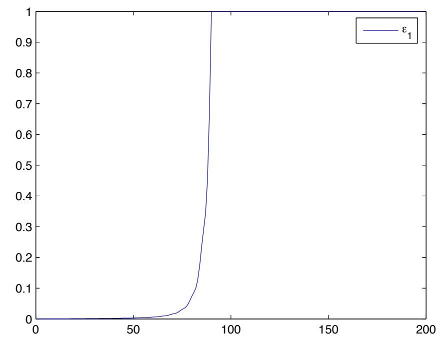


Figure 7. Time evolution of ε

7.2.1. Semi-global stabilization

- According to $\mathscr{W}$, we choose $\varepsilon_1 = 0.05$.
- According to $\mathscr{W}$ and ε_1, we choose $\varepsilon_2 = 3 \times 10^{-3}$.
- The controller is given by $u = \begin{pmatrix} -0.2674 & -0.0442 & -0.0738 & 0.0119 \end{pmatrix} \bar{x}$

The simulation data and I/O of saturation elements are shown respectively in Figure 8 and 9:

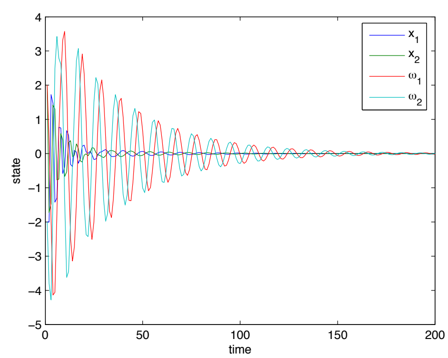
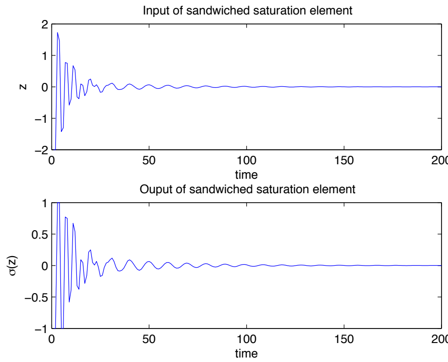


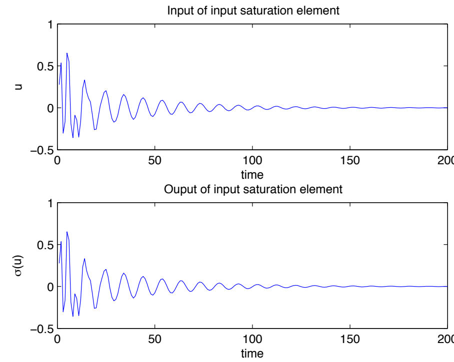
Figure 8. Semi-global stabilization of single layer sandwich system with input saturation

7.2.2. Global stabilization Note that the theoretical bounds on δ in Theorem 7 as given by (36) certainly suffice to prove solvability but it might be unnecessarily conservative in practice, since it is derived based on a worst-scenario estimation of v_1, v_2 and z_0 in (39) and (42). A proper δ can be obtained by relaxing one or more conservative bounds in (36) and reducing it again if necessary until stability is achieved as well as reasonable performance.

- We use $\delta = 0.2$. This choice is verified by a simulation of an 1296-point array of initial conditions without any observation of instability.



Sandwiched saturation



Input saturation

Figure 9. I/O of saturation elements in semi-global stabilization of single layer sandwich system with input saturation

- $M_2 = 3.7321$ and $M_3 = 6.5474$.
- The controller is formed by the semi-global stabilizing controller together with scheduling (34) and (37).

The simulation data is shown in Figure 10.

8. Conclusion

In this paper, we have considered a class of nonlinear sandwich systems, where the nonlinear element is a saturation. At first we dealt with single-layer sandwich systems, consisting of a single saturation sandwiched between two linear systems. We have established necessary and sufficient conditions for semi-global and global internal stabilization of such systems, and we have presented generalized low-gain and generalized scheduled low-gain design methodologies to achieve the prescribed stabilization. We have extended the design methodology to single-layer sandwich systems subject to input saturation, and further to multi-layer sandwich systems.

For ease of presentation, we have chosen to base the design methodologies in this paper on Riccati equations. It is also possible to generalize the classical eigenstructure assignment method from [17] to achieve the same results.

Current research is focused on constructing measurement feedback controllers to solve the semi-global and global internal stabilization problems, as well as external stabilization problems.

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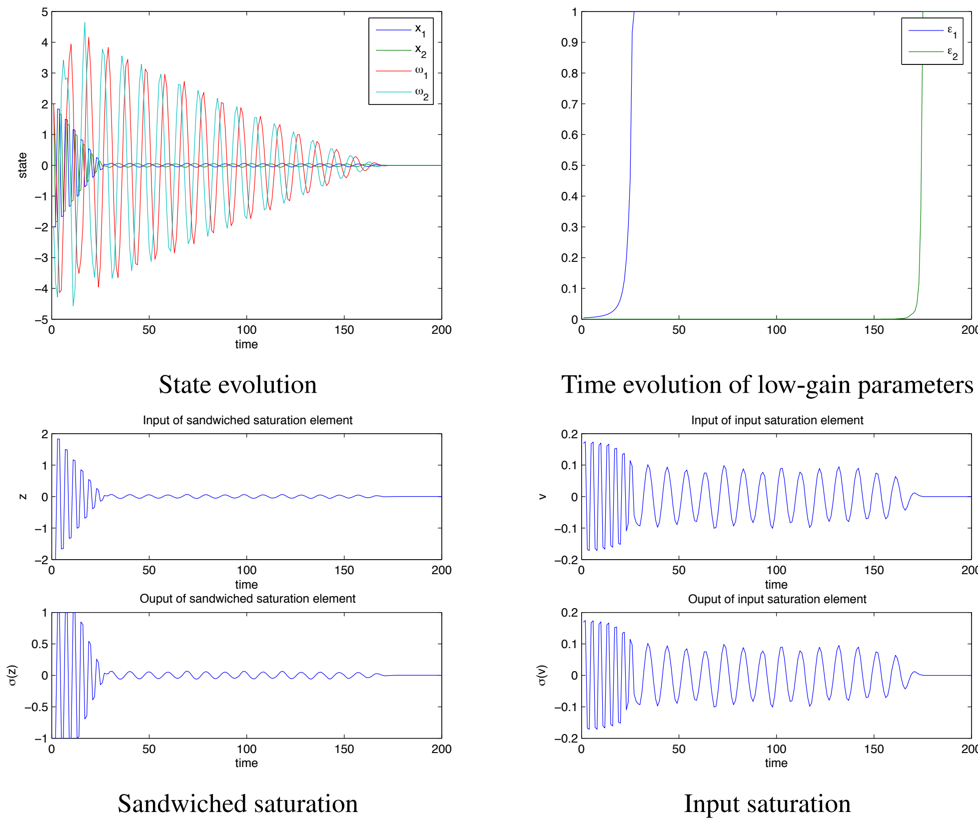

Figure 10. Global stabilization of single layer sandwich system with Input saturation

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