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Control of open-loop neutrally stable systems subject to actuator saturation and external disturbances

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SUMMARY

In this paper we study the disturbance response of open-loop neutrally stable linear systems with saturating linear feedback controller. It is shown that the closed-loop states remain bounded if the disturbances consists of those signals that do not have large sustained frequency components corresponding to the system's eigenvalues on the imaginary axis (continuous-time) or on the unit circle (discrete-time). The results in this paper are extensions of previous results by [1]. Copyright © 0000 John Wiley & Sons, Ltd.

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1. INTRODUCTION

In this paper, we study the control of open-loop neutrally stable systems subject to input saturation and external disturbances. A linear system $\rho x = Ax + Bu$ is said to be neutrally stable if A has all its eigenvalues in the closed left half plane (closed unit disc for discrete-time systems) and at least one eigenvalue on the imaginary axis (unit circle for discrete-time systems); and all the eigenvalues on the imaginary axis (unit circle for discrete-time systems) have Jordan block size 1.

In the literature dealing with external stability of linear systems subject to input saturation, the types of disturbances studied may be classified as input-additive and non-input-additive. For the input-additive case, it has been shown that there exists a linear state feedback that achieves $\mathscr{L}_p$ (continuous-time) or ℓ_p (discrete-time) stability with finite gain for $p \in [1,\infty]$ [2, 3, 4]. On the other hand, $\mathscr{L}_p$ (ℓ_p) stabilization with finite gain has been shown to be generally impossible in the non-input-additive case [5]. The authors in [6] showed that for an open-loop neutrally stable system with input saturation and non-additive disturbances, $\mathscr{L}_p$ (ℓ_p) stability *without* finite gain is attainable via a linear state feedback; however, this result only applies to $\mathscr{L}_p$ (ℓ_p) disturbances for $p \in [1,\infty)$, and not to signals belonging to $\mathscr{L}_\infty$ (ℓ_∞) space. It is also shown in [7] (see also [2]) that for continuous-time neutrally stable systems, finite gain $\mathscr{L}_\infty$ stabilization via linear state feedback is possible if the non-additive disturbances are sufficiently small. The same conclusion can be drawn for discrete-time neutrally stable system as well, following the argument used in [3].

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In continuous-time case, for signals in $\mathscr{L}_\infty$ (ℓ_∞) that are non-additive, another direction of research has focused on identifying classes of disturbances for which a controller can be designed to yield bounded closed-loop state trajectories. Work along this line has been carried out by our group in [8, 9, 1]. In that work, a set of *integral-bounded* signals has been defined as

$$S_\infty = \left\{ d \in \mathscr{L}_\infty \mid \exists M\ s.t. \forall t_2 \geq t_1 \geq 0, \left\| \int_{t_1}^{t_2} d(t)\,\mathrm{d}t \right\| \leq M \right\}.$$

The set S_∞ represents signals that have a uniformly bounded integral over every time interval; that is, signals that have no sustained DC bias. For neutrally stable systems consisting only of single integrators (i.e., eigenvalues at the origin with Jordan block size 1), it has been shown that the state trajectories remain bounded for all initial conditions and all integral-bounded disturbances. Moreover, this result also holds if we add a sufficiently small DC signal to the disturbances.

The results for continuous-time single-integrator systems appeared in the larger context of studying chains of integrators, for which it was shown that integral-bounded disturbances can be handled by an appropriately chosen control law if they are matched with the input.

In this paper we shall extend the results for single-integrator systems to neutrally stable systems. Although a similar result for discrete-time integrator-chain type system as obtained in [8, 9, 1] is not available yet, we do observe a substantial similarity between continuous- and discrete-time neutrally stable systems. The extension made to continuous-time system carries over to its discrete-time counter part. Roughly speaking, we shall show that for disturbances that do not have large sustained frequency components corresponding to the system's eigenvalues on the stability margin, a linear static state feedback can be employed to achieve boundedness of the trajectories for any initial condition and at the same time yield a globally asymptotically stable equilibrium.

2. PRELIMINARIES

2.1. *Notation*

We first recall some standard notations. $\mathbb{C}^-$ and $\mathbb{C}^\odot$ denote the open left-half complex plane and open unit disc. $\mathbb{C}^\bigcirc$ denotes the imaginary axis for continuous-time system and unit circle for discrete-time system. For $x \in \mathbb{R}^n$, $\|x\|$ denotes its Euclidean norm and x' denotes the transpose of x. For $X \in \mathbb{R}^{n\times m}$, $\|X\|$ denotes its induced 2-norm and X' denotes the transpose of X. For continuous-time (discrete-time) signal y, $\|y\|_\infty$ denotes it $\mathscr{L}_\infty$ (ℓ_∞) norm. $\mathscr{L}_\infty(\delta)$ ($\ell_\infty(\delta)$) represent a set of continuous-time (discrete-time) signals whose $\mathscr{L}_\infty$ (ℓ_∞) norm is less than δ.

2.2. *Problem formulation*

Consider the following system

$$\rho x = Ax + B\sigma(u) + Ed, \quad x(0) = x_0, \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $d \in \mathbb{R}^p$ and ρx represents $\dot{x}$ for continuous-time systems and $x(k+1)$ for discrete-time systems. $\sigma(\cdot)$ denotes the standard saturation function defined as

$$\sigma(\xi) = \begin{bmatrix} \mathrm{sign}(\xi_1)\min\{1, |\xi_1|\} \\ \vdots \\ \mathrm{sign}(\xi_m)\min\{1, |\xi_m|\} \end{bmatrix} \tag{2}$$

The pair (A,B) is stabilizable and A has all its eigenvalues in $\mathbb{C}^- \cup \mathbb{C}^\bigcirc$ for continuous-time system and $\mathbb{C}^\odot \cup \mathbb{C}^\bigcirc$ for discrete-time system, with those on $\mathbb{C}^\bigcirc$ simple. We also assume $d \in \mathscr{L}_\infty$ in the continuous-time case and $d \in \ell_\infty$ in the discrete-time case.

In the sequel, we shall identify a class of disturbances for which a properly chosen linear state feedback $u = Fx$ can be found such that the states of closed-loop system remain bounded for any

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initial condition and that in the absence of d the origin is globally asymptotically stable. Note that system (1) can be decomposed into the following form:

$$\begin{bmatrix}\rho x_s\\ \rho x_u\end{bmatrix} = \begin{bmatrix}A_s & 0\\ 0 & A_u\end{bmatrix}\begin{bmatrix}x_s\\ x_u\end{bmatrix} + \begin{bmatrix}B_s\\ B_u\end{bmatrix}\sigma(u) + \begin{bmatrix}E_s\\ E_u\end{bmatrix}d,$$

where A_s is asymptotically stable, (A_u, B_u) is controllable and A_u only has eigenvalues on $\mathbb{C}^{\bigcirc}$ with Jordan size 1. Since A_s is asymptotically stable, $d \in \mathscr{L}_\infty$ or $d \in \ell_\infty$ and $\sigma(\cdot)$ is uniformly bounded, it follows that the x_s dynamics will remain bounded no matter what controller is used. Therefore, without loss of generality, we can ignore the asymptotically stable dynamics and assume in (1) that A has eigenvalues on $\mathbb{C}^{\bigcirc}$ with Jordan size 1. Equivalently, we can assume that $A + A' = 0$ for continuous-time systems or $A'A = I$ for discrete-time systems.

To establish the results in this paper, we shall need two fundamental lemmas.

Lemma 1
Suppose $A + A' = 0$ and (A, B) is controllable. Consider the system

$$\dot{x} = Ax - B\sigma(B'x + v_1) + Bv_2, \quad x(0) = x_0$$

We have that

1. In the absence of v_1 and v_2, the origin is globally asymptotically stable;
2. $x \in \mathscr{L}_\infty$ for all $v_1 \in \mathscr{L}_\infty$, $v_2 \in \mathscr{L}_\infty(1/2)$ and any initial condition.

Lemma 2
Suppose $A'A = I$ and (A, B) is controllable. Consider

$$x(k+1) = Ax(k) - B\sigma(\kappa B'Ax(k) + v_1(k)) + Bv_2(k), \quad x(0) = x_0.$$

For κ such that $4\kappa B'B \le I$, we have

1. In the absence of v_1 and v_2, the origin is globally asymptotically stable;
2. $x \in \ell_\infty$ for all $v_1 \in \ell_\infty$, $v_2 \in \ell_\infty(1/2)$ and any initial condition.

Lemma 1 is similar to Lemma 2 in [2] and Proposition 1 in [7]. Lemma 2 basically follows from the same argument as used in proof of Proposition 2.3 in [4]. The detailed proofs are appended at the end of the paper.

3. CONTINUOUS-TIME SYSTEMS

We first study a continuous-time system

$$\dot{x} = Ax + B\sigma(u) + Ed$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $d \in \mathbb{R}^p$. Also assume that (A, B) is controllable, $A + A' = 0$ and $d \in \mathscr{L}_\infty$.

We employ a linear static state feedback $u = -B'x$, which results in a closed-loop system

$$\dot{x} = Ax - B\sigma(B'x) + Ed, \quad x(0) = x_0. \tag{3}$$

Global asymptotic stability follows from Lemma 1. We focus here only on the boundedness of the closed-loop states.

3.1. Extended class of disturbances

To present our results, we extend the definition of integral-bounded disturbances introduced in [8, 9, 1] by defining a new set

$$\Omega_\infty = \{d \in \mathscr{L}_\infty \mid \forall i \in 1, \ldots, q,\ d(t)\sin\omega_i t \in S_\infty \text{ and } d(t)\cos\omega_i t \in S_\infty\}, \tag{4}$$

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where $\pm j\omega_i$, $i \in 1,\ldots,q$, represents the eigenvalues of A. The set Ω_∞ consists of those signals that remain integral-bounded when multiplied by $\sin\omega_i t$ and $\cos\omega_i t$. This definition is a natural generalization of S_∞, since $\Omega_\infty = S_\infty$ for $\omega_i = 0$ in a chain of integrators.

In practical terms, a signal that belongs to Ω_∞ is a signal that has no sustained frequency component at any of the frequencies ω_i, $i \in 1,\ldots,q$. To see this, note that we can equivalently write

$$\Omega_\infty = \left\{ d \in \mathscr{L}_\infty \mid \exists M \;\; s.\,t.\; \forall i \in 1,\ldots,q, \forall t_2 \geq t_1 \geq 0, \left\| \int_{t_1}^{t_2} d(t)\mathrm{e}^{j\omega_i t}\,\mathrm{d}t \right\| \leq M \right\}. \tag{5}$$

The integral $\int_{t_1}^{t_2} d(t)\mathrm{e}^{j\omega_i t}\,\mathrm{d}t$ is easily recognized as the value at ω_i of the Fourier transform of the signal $d(t)$ truncated to the interval $[t_1, t_2]$. The definition of Ω_∞ implies that this value must be uniformly bounded regardless of the choice of t_1 and t_2.

In tune with the results for the single-integrator case, we shall show in the following sections that the trajectories of the controlled system (3) remain bounded for all disturbances belonging to Ω_∞. Moreover, this result also holds if we add a sufficiently small signal that does not belong to Ω_∞.

3.2. *Second order single-frequency system*

We start by considering an example system with a pair of complex eigenvalues at $\pm j$:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} - \begin{bmatrix}0\\ 1\end{bmatrix}\sigma(x_2) + \begin{bmatrix}e_1\\ e_2\end{bmatrix} d, \quad \begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix} = x_0. \tag{6}$$

Theorem 1
The trajectories of (6) remain bounded for any $d \in \Omega_\infty$ and any x_0.

Proof
To analyze the system, we start by introducing a rotation matrix

$$R = \begin{bmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{bmatrix},$$

which represents a counterclockwise rotation by an angle t. The dynamics of the rotation matrix is given by

$$\dot{R} = -R\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}.$$

We shall study the dynamics of x from a rotated coordinate frame, and toward this end we define the rotated state $y = Rx$. The dynamics of y is given by

$$\begin{aligned}
\dot{y} &= \dot{R}x + R\dot{x}\\
&= R\left(\begin{bmatrix}e_1\\ e_2\end{bmatrix} d - \begin{bmatrix}0\\ 1\end{bmatrix}\sigma(x_2)\right)\\
&= R\left(\begin{bmatrix}e_1\\ e_2\end{bmatrix} d - \begin{bmatrix}0\\ 1\end{bmatrix}\sigma\left(\begin{bmatrix}0 & 1\end{bmatrix}R'y\right)\right), \quad y(0) = x(0) = x_0.
\end{aligned}$$

Next, define a fictitious system

$$\dot{\tilde{y}} = R\begin{bmatrix}e_1\\ e_2\end{bmatrix} d, \quad \tilde{y}(0) = x_0. \tag{7}$$

We know from the definition of Ω_∞ that the signal $d(t)$ is integral-bounded when multiplied by $\sin t$ and $\cos t$. It therefore follows that the right-hand side of (7) is integral-bounded, and hence $\tilde{y} \in \mathscr{L}_\infty$.

Consider the difference between y and the fictitious state $\tilde{y}$, given by $z = y - \tilde{y}$, with dynamics

$$\begin{aligned}\dot{z} &= -R\begin{bmatrix}0\\1\end{bmatrix}\sigma(\begin{bmatrix}0 & 1\end{bmatrix}R'y)\\ &= -R\begin{bmatrix}0\\1\end{bmatrix}\sigma(\begin{bmatrix}0 & 1\end{bmatrix}R'z+\delta), \quad z(0)=0\end{aligned}$$

where $\delta = [0,1]R'\tilde{y} \in \mathscr{L}_\infty$. We rotate z back to the original coordinate frame by introducing $w = R'z$, thereby obtaining the dynamics

$$\begin{aligned}\dot{w} &= \dot{R}'z + R'\dot{z}\\ &= \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}w - \begin{bmatrix}0\\1\end{bmatrix}\sigma(\begin{bmatrix}0 & 1\end{bmatrix}w+\delta), \quad w(0)=0.\end{aligned}$$

It follows from Lemma 1 that $w \in \mathscr{L}_\infty$. Finally, we have $x = w + R'\tilde{y}$, and hence $x \in \mathscr{L}_\infty$. □

To demonstrate the importance of the disturbance belonging to Ω_∞, we shall now show that if d contains a large frequency component at $\pm j$, the states of (6) will diverge toward infinity for any initial condition. Suppose therefore that $d(t) = a\sin(t+\theta)$, where a is an amplitude yet to be chosen. For ease of presentation, we assume that $[e_1, e_2]' = [0,1]'$. Consider the dynamics of the rotated state y from the proof of Theorem 1. We have

$$\begin{aligned}\dot{y} &= R\begin{bmatrix}0\\1\end{bmatrix}\left(d - \sigma(\begin{bmatrix}0 & 1\end{bmatrix}R'y)\right)\\ &= a\begin{bmatrix}-\sin t\sin(t+\theta)\\ \cos t\sin(t+\theta)\end{bmatrix} + \begin{bmatrix}\sin t\\ -\cos t\end{bmatrix}\sigma(\cdot).\end{aligned}$$

Using trigonometric identities, the dynamics can be rewritten as

$$\dot{y} = \frac{a}{2}\begin{bmatrix}\cos(2t+\theta)-\cos(\theta)\\ \sin(2t+\theta)+\sin(\theta)\end{bmatrix} + \begin{bmatrix}\sin t\\ -\cos t\end{bmatrix}\sigma(\cdot).$$

We have that either $|\sin(\theta)| \geq \sqrt{2}/2$ or $|\cos(\theta)| \geq \sqrt{2}/2$. Without loss of generality, we assume $|\sin(\theta)| \geq \sqrt{2}/2$. Let a be chosen such that $a \geq 4/\sqrt{2}(1+\varepsilon)$, where ε is a positive number. For the trajectory $y_2(t)$, we have

$$|y_2(t)| = \left|y_2(0) + \int_0^t \frac{a}{2}[\sin(2\tau+\theta)+\sin(\theta)] - \cos\tau\sigma(\cdot)\,\mathrm{d}\tau\right|.$$

Noting that the last term of the integrand is bounded by ± 1, and using the bound $|a/2\sin(\theta)| \geq \sqrt{2}a/4 \geq 1+\varepsilon$, we therefore have

$$\begin{aligned}|y_2(t)| &\geq \int_0^t \varepsilon\,\mathrm{d}\tau - |y_2(0)| - \frac{a}{2}\left|\int_0^t \sin(2\tau+\theta)\,\mathrm{d}\tau\right|\\ &\geq \varepsilon t - |y_2(0)| - \frac{a}{2}.\end{aligned}$$

This shows that $y_2(t)$ diverges toward infinity.

3.3. Connection to single-integrator case

Before moving on to the case of general multi-frequency systems, it is instructive to compare some aspects of the above example with previous results for single-integrator systems. A single-integrator system with a saturated control input and an external disturbance has the form

$$\dot{x} = \sigma(\cdot) + ed.$$

In the absence of disturbances, the open-loop response of this system is stationary. It is intuitively easy to see that a large DC bias in d would constitute a problem, because it would tend to dominate the bounded control term $\sigma(\cdot)$, thus leading to unboundedness. The absence of such a DC bias is guaranteed by d belonging to S_∞.

The example system above has the form

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sigma(\cdot) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} d.$$

In the absence of disturbances, the open-loop response of this system is oscillatory rather than stationary, and it is less obvious why a disturbance that does not belong to Ω_∞ may be problematic. By introducing a rotated state $y = Rx$, however, we obtain the dynamics

$$\dot{y} = R \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sigma(\cdot) + R \begin{bmatrix} 0 \\ 1 \end{bmatrix} d.$$

In the absence of disturbances, the open-loop response of y is stationary, and the dynamics of y are similar to the single-integrator case. In particular, it is easy to see that a large DC bias in the term $R\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]d$ would constitute a problem, because it would tend to dominate the bounded control term. Analogous to the single-integrator case, the absence of such a bias is guaranteed if $R\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]d$ belongs to S_∞, which is equivalent to d belonging to Ω_∞.

In the single-integrator case, a DC bias in d can be tolerated if it is sufficiently small. Similarly, a small signal that does not belong to Ω_∞ can be tolerated for systems with complex eigenvalues. This is demonstrated in the next section, which deals with general multi-frequency systems.

3.4. Multi-frequency systems

We first extend Theorem 1 to a multi-frequency neutrally stable system. Consider

$$\dot{x} = Ax - B\sigma(B'x) + Ed, \quad x(0) = x_0 \tag{8}$$

where $A + A' = 0$ and (A, B) is controllable. Without loss of generality, we assume that

$$A = \begin{bmatrix} A_1 & & & \\ & \ddots & & \\ & & A_s & \\ & & & 0 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_s \\ x_o \end{bmatrix} \tag{9}$$

where $x_i \in \mathbb{R}^2$, $i = 1, \ldots, s$, $x_o \in \mathbb{R}^{n-2s}$ and

$$A_i = \begin{bmatrix} 0 & \omega_i \\ -\omega_i & 0 \end{bmatrix}, \quad i = 1, \ldots, s$$

with $2s \leq n$. We have the following theorem

Theorem 2
The states of (8) remain bounded for any $d \in \Omega_\infty$ and any x_0.

Proof
Consider the matrix

$$R = \begin{bmatrix} R_1 & & & \\ & \ddots & & \\ & & R_s & \\ & & & I \end{bmatrix}$$

where

$$R_i = \begin{bmatrix} \cos \omega_i t & -\sin \omega_i t \\ \sin \omega_i t & \cos \omega_i t \end{bmatrix}.$$

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Note that R is unitary, i.e. $RR' = I$ and moreover

$$\dot{R} = -RA.$$

Define a transformed state $y = Rx$. As a result,

$$\dot{y} = -RB'\sigma(B'R'y) + REd, \quad y(0) = x_0.$$

Introduce a fictitious system

$$\dot{\tilde{y}} = REd, \quad \tilde{y}(0) = x_0.$$

It follows from the definition of Ω_∞ that $\tilde{y} \in \mathscr{L}_\infty$. Next, define the difference between y and $\tilde{y}$ by $z = y - \tilde{y}$. We get

$$\dot{z} = -RB\sigma(B'R'z + B'R'\tilde{y}), \quad z(0) = 0.$$

Finally transform z back to the original coordinates by defining $w = R'z$. Note that

$$\dot{R}' = AR'.$$

Hence

$$\dot{w} = Aw - B\sigma(B'w + B'R'\tilde{y}), \quad \tilde{w}(0) = 0.$$

Lemma 1 shows that $w \in \mathscr{L}_\infty$. Since $x = w + R'\tilde{y}$ and $\tilde{y} \in \mathscr{L}_\infty$ is bounded for all t, we conclude that $x \in \mathscr{L}_\infty$. □

Next, we shall prove that the states of (8) also remain bounded if a small signal that does not belong to Ω_∞ is added on top of the original signal in Ω_∞. Consider the system

$$\dot{x} = Ax - B\sigma(B'x) + E_1 d_1 + E_2 d_2, \quad x(0) = x_0 \tag{10}$$

where $A + A' = 0$ and (A, B) is controllable. Without loss of generality we assume that A is in the form of (9).

Theorem 3
The states of system (10) remain bounded for any x_0, any $d_1 \in \Omega_\infty$ and $d_2 \in \mathscr{L}_\infty(\delta)$ with δ sufficiently small..

Proof
Using the same sequence of transformations as introduced in the proof of Theorem 2, we get the following transformed system

$$\dot{w} = Aw - B\sigma(B'w + B'R'\tilde{y}) + E_2 d_2, \quad w(0) = 0$$

where $w = x - R'\tilde{y}$ and

$$\dot{\tilde{y}} = RE_1 d_1, \quad \tilde{y} = x_0.$$

The fact that $d_1 \in \Omega_\infty$ implies that $\tilde{y} \in \mathscr{L}_\infty$. Introduce another fictitious system

$$\dot{\bar{w}} = (A - BB')\bar{w} + E_2 d_2, \quad \bar{w}(0) = 0.$$

Since $A - BB'$ is Hurwitz stable and $d_2 \in \mathscr{L}_\infty(\delta)$, we have that $\bar{w} \in \mathscr{L}_\infty$ and moreover $\|B'\bar{w}\|_\infty \leq \frac{1}{2}$ for sufficiently small δ.

Define $\tilde{w} = w - \bar{w}$. Then $\tilde{w}$ has the following dynamics

$$\dot{\tilde{w}} = A\tilde{w} - B\sigma(B'\tilde{w} + v_1) + Bv_2, \quad \tilde{w} = 0$$

where $v_1 = B'\bar{w} + B'R'\tilde{y}$ and $v_2 = B'\bar{w}$. It follows from Lemma 1 that $\tilde{w} \in \mathscr{L}_\infty$. Since $x = \bar{w} + \tilde{w} + R'\tilde{y}$ and $\bar{w}, R'\tilde{y} \in \mathscr{L}_\infty$, we conclude that $x \in \mathscr{L}_\infty$. □

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4. DISCRETE-TIME SYSTEMS

In this section, we deal with discrete-time systems. Consider the following system

$$x(k+1) = Ax(k) + B\sigma(u(k)) + Ed(k), \quad x(0) = x_0. \tag{11}$$

We assume that (A,B) is controllable and $A'A = I$.

We use a linear state feedback controller $u = -\kappa B'Ax$ which gives a closed-loop system as

$$x(k+1) = Ax(k) + B\sigma(-\kappa B'Ax(k)) + Ed(k), \quad x(0) = x_0.$$

For κ such that $4\kappa B'B \leq I$, the global asymptotic stability of the origin in the absence of d follows from Lemma 2. As such, as in continuous-time case, we focus here only on the boundedness of closed-loop states with disturbances.

4.1. Extended class of disturbances

As in continuous-time case, we define a set of discrete disturbances

$$\Omega_\infty = \Bigg\{ d \in \ell_\infty \mid \exists M > 0,\ s.\,t. \forall i \in 1,\ldots,q,\ \forall k_2 \geq k_1 \geq 0, \\ \left\| \sum_{k=k_1}^{k_2} d(k)\cos(\theta_i k) \right\| \leq M, \quad \left\| \sum_{k=k_1}^{k_2} d(k)\sin(\theta_i k) \right\| \leq M \Bigg\}, \tag{12}$$

where $e^{j\theta_i}$, $i \in 1,\ldots,q$, represents the eigenvalues of A.

Ω_∞ contains signals which do not have sustained component at discrete frequency θ_i. Like in the continuous-time case, we can also rewrite the above definition as

$$\Omega_\infty = \left\{ d \in \ell_\infty \mid \exists M > 0,\ s.\,t. \forall i \in 1,\ldots,q,\ \forall k_2 \geq k_1 \geq 0, \quad \left\| \sum_{k=k_1}^{k_2} d(k) z_i^k \right\| \leq M \right\}, \tag{13}$$

where $z_i = e^{j\theta_i}$, $i = 1,\ldots,q$, denotes the eigenvalues of A. Since $d \in \ell_\infty$, the power series $\sum_0^\infty d(z)z^k$ or the z-transform of $d(k)$ always has a radius of convergence 1. On $|z| = 1$, definition (12) implies all partial sums of the power series is bounded at $z = z_i$.

Note that the set Ω_∞ in (12) and (13) are a discrete equivalent of (4) and (5).

4.2. Multi-frequency systems

Next we shall prove the boundedness of closed-loop trajectories with disturbances that belong to Ω_∞ as defined in (12). The philosophy of the proof is basically the same as in continuous-time case. We apply a sequence of successive rotations to state coordinates and eventually convert the non-input-additive disturbances to input-additive disturbances using the property of Ω_∞. Since this procedure has been made clear in the preceding section, we shall skip the proof for second-order single-frequency systems and only work on the general case.

Theorem 4

Consider the system

$$x(k+1) = Ax(k) - B\sigma(\kappa B'Ax(k)) + Ed(k), \quad x(0) = x_0 \tag{14}$$

where (A,B) is controllable, $A'A = I$ and $d \in \Omega_\infty$. Then for κ such that $4\kappa B'B \leq I$, we have $x(k)$ bounded for all $k \geq 0$ and for any initial condition.

Proof

Define $R(k) = (A')^k$. Since $A'A = I$, $R(k)$ represents a time-varying rotation matrix with difference

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equation
$$R(k+1) = R(k)A'$$
Also define
$$y(k) = R(k)x(k)$$
The transformed system becomes
$$y(k+1) = y(k) - R(k)A'B\sigma(\kappa B'AR'(k)y(k)) + R(k)A'Ed(k), \quad y(0) = x_0.$$
Introduce a fictitious system
$$\tilde{y}(k+1) = \tilde{y}(k) + R(k)A'Ed(k), \quad \tilde{y}(0) = x_0.$$
Note that $d \in \Omega_\infty$ implies that there exists a $M > 0$ such that
$$\forall k_2 > k_1 > 0, \left\| \sum_{k_1}^{k_2} (A')^k Ed(k) \right\| \leq M.$$
Therefore, we find $\tilde{y} \in \ell_\infty$. Let $z = y - \tilde{y}$. We get
$$z(k+1) = z(k) - R(k)A'B\sigma(\kappa B'AR'(k)z(k) + \kappa B'AR'(k)\tilde{y}(k)), \quad z(0) = 0.$$
Finally, define $w(k) = R'(k)z(k)$. The dynamics of w is given by
$$w(k+1) = Aw(k) - B\sigma(\kappa B'Aw(k) + v(k)), \quad w(0) = 0.$$
where $v(k) = \kappa B'A^{k+1}\tilde{y}(k)$. It follows from Lemma 2 that the above system is ℓ_∞ stable with respect to v given $4\kappa B'B \leq I$. Thus $\tilde{y}_\infty$ implies $w \in \ell_\infty$. Note that $x(k) = w(k) + R'(k)\tilde{y}(k)$. Therefore, we conclude $x \in \ell_\infty$. □

The next theorem shows that a small disturbance that does not belong to Ω_∞ can also be tolerated.

Theorem 5
Consider the discrete-time system
$$x(k+1) = Ax(k) - B\sigma(\kappa B'Ax(k)) + E_1 d_1(k) + E_2 d_2(k), \quad x(0) = x_0. \tag{15}$$
For κ such that $4\kappa B'B \leq I$, we have $x(k)$ bounded for all $k \geq 0$ and for any x_0, $d_1 \in \Omega_\infty$ and $d_2 \in \ell_\infty(\delta)$ with δ sufficiently small.

Proof
Following the same lines as in the proof of Theorem 4, we shall get a transformed system
$$w(k+1) = Aw(k) - B\sigma(\kappa B'Aw(k) + \kappa B'AR'(k)\tilde{y}(k)) + E_2 d_2(k), w(k) = 0$$
where $w(k) = x(k) - R'(k)\tilde{y}(k)$ and $\tilde{y}$ satisfies
$$\tilde{y}(k+1) = \tilde{y}(k) + R(k)A'E_1 d_1(k), \quad \tilde{y}(0) = x_0,$$
and hence $\tilde{y} \in \ell_\infty$. Introduce an auxiliary system
$$\bar{w}(k+1) = (A + BF)\bar{w}(k) + E_2 d_2(k), \quad \bar{w}(0) = 0,$$
where F is such that $A + BF$ is asymptotically stable. Since $d_2 \in \ell_\infty(\delta)$, we find that $F\bar{w} \in \ell_\infty$. Moreover, we have $\|F\bar{w}\|_\infty \leq 1/2$ with sufficiently small δ.

Define $\tilde{w} = w - \bar{w}$. Then we get
$$\tilde{w}(k+1) = A\tilde{w}(k) - B\sigma(\kappa B'A\tilde{w}(k) + v_1(k)) + Bv_2(k), \quad \tilde{w}(0) = 0.$$
where $v_1 = \kappa B'A\bar{w} + \kappa B'A^{k+1}\tilde{y}$ and $v_2 = F\bar{w}$.

Lemma 2 shows that $\tilde{w} \in \ell_\infty$. Since $x = \tilde{w} + \bar{w} + R'(k)\tilde{y}$, we conclude that $x \in \ell_\infty$ for any initial condition. □

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5. CONCLUSION

In this paper, we study the dynamics of an open-loop neutrally stable linear system controlled by a saturating linear feedback controller in the presence of external disturbances. Two classes of disturbances have been identified for which we can achieve bounded states of the closed-loop system. This paper extends the results for a single-integrator system as reported in [1] to a neutrally stable system. It is evident that the class of disturbances identified in this paper is a natural extension of the class of integral-bounded disturbances for a chain of integrator. The more general case of critically unstable systems which have complex eigenvalues with Jordan block size great than 1 is subject to current research.

APPENDIX

We shall develop proof for Lemma 1 and 2. In order to do so, we need the following inequalities, which were proven in [6]:

Lemma 3
For two vectors $u, w \in \mathbb{R}^m$, the following statements hold:

$$|u'[\sigma(u+w)-\sigma(u)]| \leq 2\sqrt{m}\|w\|; \tag{16}$$

$$2u'[\sigma(w)-\sigma(w-u)] \geq u'\sigma(u), \quad \|w\| \leq \tfrac{1}{2}; \tag{17}$$

$$\|u-\sigma(u)\| \leq u'\sigma(u); \tag{18}$$

$$-u'[\sigma(u)+w] \leq \tfrac{\|w\|^2}{4}, \quad \|w\| \leq 1, \tag{19}$$

where $\sigma(\cdot)$ is the standard saturation function defined in (2).

Proof of Lemma 1
Item 1 has been proven in [2]. We only prove item 2. Denote $u = B'x$ and define $V_1 = \frac{1}{3}\|x\|^3$. Differentiating V_1 along the trajectories yields

$$\begin{aligned}
\dot{V}_1 &= \|x\|u'\left[\sigma(-u+v_1)+v_2\right] \\
&\leq \|x\|(u-v_1)'\left[-\sigma(u-v_1)+v_2\right]+2\|x\|\|v_1\|_\infty \\
&= \|x\|\left\{(u-v_1)'[-\sigma(u-v_1)+\sigma(u-v_1+v_2)]\right. \\
&\quad \left.+(u-v_1)'[-\sigma(u-v_1+v_2)+\sigma(v_2)]\right\}+2\|x\|\|v_1\|_\infty \\
&\leq -\tfrac{1}{2}\|x\|(u-v_1)\sigma(u-v_1)+2\sqrt{m}\|x\|\|v_2\|_\infty+2\|x\|\|v_1\|_\infty.
\end{aligned}$$

The last inequality results from (16), (17) and the condition $\|v_2\| \leq \frac{1}{2}$.
Next, since $A-BB'$ is Hurwitz stable, there exists a $P>0$ satisfying

$$(A-BB')'P+P(A-BB') = -I.$$

Define $V_2 = x'Px$. There exists an α such that

$$\begin{aligned}
\dot{V}_2 &= -\|x\|^2+2x'P[B\sigma(-u+v_1)+Bu+Bv_2] \\
&= -\|x\|^2+2x'P[B(\sigma(-u+v_1)+u-v_1)+Bv_2+Bv] \\
&\leq -\|x\|^2+2\alpha\|x\|(u-v_1)\sigma(u-v_1)+2\alpha\|x\|\|v_2\|_\infty+2\alpha\|x\|\|v_1\|_\infty,
\end{aligned}$$

where inequality (18) is used to derive the last inequality.
Finally, define a Lyapunov candidate $V = 4\alpha V_1+V_2$. We find that

$$\begin{aligned}
\dot{V} &\leq -\|x\|^2+(8\alpha\sqrt{m}+2\alpha)\|x\|\|v_2\|_\infty+10\alpha\|x\|\|v_1\|_\infty \\
&= -\|x\|\left[\|x\|-(8\alpha\sqrt{m}+2\alpha)\|v_2\|_\infty-10\alpha\|v_1\|_\infty\right].
\end{aligned}$$

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Hence $\dot{V} \le 0$ for $\|x\| \ge (8\alpha\sqrt{m}+2\alpha)\|v_2\|_\infty + 10\alpha\|v_1\|_\infty$. Let c be such that

$$\{x \mid V(x) \le c\} \supset \{x \mid \|x\| \le (8\alpha\sqrt{m}+2\alpha)\|v_2\|_\infty + 10\alpha\|v_1\|_\infty\}.$$

We have $\dot{V} \le 0$ for $x \notin \{x \mid V(x) \le c\}$. This implies that $x(t) \in \{x \mid V(x) \le c\}$ for all $t \ge 0$. □

To prove Lemma 2, we borrow the next lemma from [6]

Lemma 4
Assume that $A'A = I$ and $\kappa B'B \le 2I$ for some $\kappa > 0$ Then $\tilde{A} = A - \kappa BB'A$ is Schur stable if and only if (A,B) is controllable.

Proof of Lemma 2
In order the prove the above result, we first need the following lemma.
Denote $\kappa B'Ax$ by u. Define $V_1 = \|x\|^2$. We have that

$$\begin{aligned}
V_1(k+1) - V_1(k) &= \|Ax + B\sigma(-u+v_1) + Bv_2\|^2 - \|x\|^2 \\
&= \tfrac{2}{\kappa}u'[\sigma(-u+v_1)+v_2] + \left[\sigma(-u+v_1)' + v_2'\right] B'B \left[\sigma(-u+v_1)' + v_2'\right] \\
&\le \tfrac{2}{\kappa}[u-v_1]'[\sigma(-u+v_1)+v_2] + \tfrac{2}{\kappa}v_1'[\sigma(-u+v_1)+v_2] + \tfrac{1}{4\kappa}\|\sigma(-u+v_1)+v_2\|^2
\end{aligned}$$

where we use condition $4\kappa BB' \le I$. Since $\|v_2\| \le \frac{1}{2}$ and $\sigma(\cdot)$ is bounded by ± 1, we find that $v_1'[\sigma(-u+v_1)+v_2] \le 2\|v_1\|$. This yields that

$$\begin{aligned}
V_1(k+1) - V_1(k) &\le \tfrac{2}{\kappa}[u-v_1]'[\sigma(-u+v_1)+v_2] + \tfrac{1}{2\kappa}\|\sigma(-u+v_1)\|^2 + \tfrac{1}{2\kappa}\|v_2\|^2 + \tfrac{4}{\kappa}\|v_1\| \\
&\le \tfrac{2}{\kappa}[u-v_1]'[\sigma(-u+v_1)+v_2] + \tfrac{1}{2\kappa}\|\sigma(-u+v_1)\|^2 + \tfrac{1}{2\kappa}\|v_2\| + \tfrac{4}{\kappa}\|v_1\|.
\end{aligned}$$

Note that

$$\begin{aligned}
\tfrac{2}{\kappa}[u-v_1]'[\sigma(-u+v_1)+v_2] &= \tfrac{1}{\kappa}[u-v_1]'\sigma(-u+v_1) + \tfrac{1}{\kappa}[u-v_1]'[\sigma(-u+v_1)+2v_2] \\
&\le \tfrac{1}{\kappa}[u-v_1]'\sigma(-u+v_1) + \tfrac{1}{\kappa}\|v_2\|^2 \\
&\le \tfrac{1}{\kappa}[u-v_1]'\sigma(-u+v_1) + \tfrac{1}{\kappa}\|v_2\|
\end{aligned}$$

where we use (19) and $\|v_2\| \le \frac{1}{2}$. Therefore,

$$\begin{aligned}
V_1(k+1) - V_1(k) &\le \tfrac{1}{\kappa}[u-v_1]'\sigma(-u+v_1) + \tfrac{1}{2\kappa}\|\sigma(-u+v_1)\|^2 + \tfrac{3}{2\kappa}\|v_2\| + \tfrac{4}{\kappa}\|v_1\| \\
&\le \tfrac{1}{2\kappa}[u-v_1]'\sigma(-u+v_1) + \tfrac{3}{2\kappa}\|v_2\| + \tfrac{4}{\kappa}\|v_1\|. \qquad (20)
\end{aligned}$$

Since $4\kappa B'B \le I$, $\tilde{A} = A - \kappa BB'A$ is Schur stable. Let P be the solution to the Lyapunov equation

$$\tilde{A}'P\tilde{A} - P + I = 0.$$

Define $V_2 = \|P^{1/2}x\|$. We have

$$\begin{aligned}
V_2(k+1) - V_2(k) &= \|P^{1/2}\tilde{A}x + P^{1/2}B[u - v_1 - \sigma(u-v_1) + (v_2+v_1)]\| - \|P^{1/2}x\| \\
&\le \|P^{1/2}\tilde{A}x\| + \|P^{1/2}B[u - v_1 - \sigma(u-v_1) + (v_2+v_1)]\| - \|P^{1/2}x\|.
\end{aligned}$$

For $x \neq 0$, there exists a $\beta > 0$ such that

$$\|P^{1/2}\tilde{A}x\| - \|P^{1/2}x\| = \frac{\|P^{1/2}\tilde{A}x\|^2 - \|P^{1/2}x\|^2}{\|P^{1/2}\tilde{A}x\| + \|P^{1/2}x\|} = \frac{-\|x\|^2}{\|P^{1/2}\tilde{A}x\| + \|P^{1/2}x\|} \le -\beta\|x\|.$$

Obviously, the above also holds for $x = 0$. Hence

$$\begin{aligned}
V_2(k+1) - V_2(k) &\le -\beta\|x\| + \|P^{1/2}B\|\|(u-v_1) - \sigma(u-v_1)\| + \|P^{1/2}B\|(\|v_2\| + \|v_1\|) \\
&\le -\beta\|x\| + \|P^{1/2}B\|(u-v_1)'\sigma(u-v_1) + \|P^{1/2}B\|(\|v_2\| + \|v_1\|) \qquad (21)
\end{aligned}$$

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where we use (18) of Lemma 3.

Define $V = 2\kappa\|P^{1/2}B\|V_1 + V_2$. We obtain from (20) and (21) that

$$V(k+1) - V(k) \leq -\beta\|x\| + 9\|P^{1/2}B\|\|v_1\| + 4\|P^{1/2}B\|\|v_2\|. \tag{22}$$

This immediately implies that $x \in \ell_\infty$ for any initial condition. □

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