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Control of a chain of integrators subject to actuator saturation and disturbances

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SUMMARY

In this paper, we study control of a chain of integrators under actuator saturation and non-additive disturbances. We shall show that boundedness of the states can be ensured if the disturbances are matched and integral-bounded; misaligned and magnitude-bounded; or a combination of the two, using either a static or a dynamic low-gain state feedback. This result is an extension of our earlier work in [19].

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KEY WORDS: chain of integrators, non-additive disturbances, actuator saturation, low-gain feedback

1. INTRODUCTION

In this paper, we study the response of a saturating feedback-controlled chain of integrators to non-additive disturbances. Specifically, we are interested in the following system:

$$\dot{x} = Ax + B\sigma(u) + Ed, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}, \quad d \in \mathbb{R}^m, \tag{1}$$

where x, A and B are given by

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}, \tag{2}$$

and u is a control input. The function $\sigma(\cdot)$ denotes a standard saturation; that is, $\sigma(u) = \text{sign}(u)\min\{1, |u|\}$. If $E = B$, the disturbance is said to be *matched* with the control input. Otherwise, the disturbance is said to be *unmatched*. We will also deal specifically with the situation $B'E = 0$, in which case we say that the disturbance is *misaligned* with the input.

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The goal is to identify a class of disturbances for which the boundedness of the state can be ensured by a static or dynamic feedback controller.

Control of linear systems subject to actuator saturation has been an area of active research over the past two decades, due to the recognition of saturation as one of the most ubiquitous and inherent physical constraints in control systems; see, for instance, [1, 4, 5, 10, 16] and references therein. For an integrator chain of order $n = 2$, it is well known that global stability can be achieved by any linear feedback that stabilizes the corresponding unsaturated system. For $n > 2$, however, it was shown in [3] that global stabilization requires nonlinear feedback. Global stabilization can in this case be achieved by either a nonlinear control law using nested saturations [17], or by a low-gain nonlinear controller with gain scheduling (see [9]).

1.1. Response to disturbances

Disturbances and uncertainties are inevitable in control engineering applications. Nevertheless, the disturbance response of a chain of integrators with saturating feedback is still not completely understood. Some previous work has been done for the double integrator (i.e., $n = 2$) with a linear static state feedback $u = Fx$ [2, 15, 11, 20, 19]. In particular, $\mathcal{L}_p$ stabilization was studied in [2, 15] and proven to be impossible for any $p > 2$. Furthermore, it was shown in [11] that input-to-state stability (ISS) (see [13]) cannot be achieved with respect to matched disturbances. In fact, for any linear feedback a disturbance with arbitrary small $\mathcal{L}_\infty$ norm and suitable initial conditions can be found such that the states of the double integrator diverge. This negative result was later extended in [20] to an even more restricted class of disturbances. However, it was also shown that boundedness of the states can be preserved for arbitrary initial condition if the disturbance has a uniformly bounded integral over all time intervals. To be precise, for any $M > 0$, we define a parameterized set of disturbances as

$$\bar{\Omega}_M = \left\{ d \in \mathcal{L}_\infty(1) \;\middle|\; \left| \int_{t_1}^{t_2} d(t)\,\mathrm{d}t \right| \leq M, \quad \forall\, t_2 \geq t_1 \geq 0 \right\},$$

where $\mathcal{L}_\infty(D)$ denotes the set of $\mathcal{L}_\infty$ signals of $\mathcal{L}_\infty$ norm less than D. For a given linear state feedback, it was proven that for any $M > 0$ there exists a $q^* > 0$ such that the states of the double integrator remain bounded for arbitrary initial condition and any disturbance $d(t) = qs(t)$, where $s(t) \in \bar{\Omega}_M$ and $q \leq q^*$. The magnitude restriction in $\bar{\Omega}_M$ and the attenuation factor q were later removed in [19], where a broader class of *integral-bounded* disturbances was defined for any $M > 0$ as follows

$$\Omega_M = \left\{ d \in \mathcal{L}_\infty \;\middle|\; \left| \int_{t_1}^{t_2} d(t)\,\mathrm{d}t \right| \leq M, \quad \forall\, t_2 \geq t_1 \geq 0 \right\}.$$

It was shown that for a priori known constants $M > 0$ and $D > 0$, the states of a double integrator controlled by a properly designed saturating linear static state feedback remain bounded for any initial condition if $d \in \Omega_M$ for the matched case and $d \in \mathcal{L}_\infty(D)$ for the misaligned case (see Theorem 2 and Lemma 3 in [19]).

1.2. Contributions

This work is a further extension of the results in [11, 20, 19] for $n > 2$. To the best of our knowledge, the only available result in the literature that is related and may be applied to the problems studied in this paper is in [14] where the authors show that for a general linear system in the form of (1), $\mathcal{L}_p$ stabilization without finite gain can be achieved for $p \in [1, \infty)$. However, for sustained disturbances that are in $\mathcal{L}_\infty$ space, the problem remains unsolved. In this paper, we shall show that a result similar to the double-integrator case holds for the case $n > 2$ as well; namely, that by the proper choice of feedback law, boundedness of the states can be ensured for both (i) matched, integral-bounded disturbances; (ii) misaligned, magnitude-bounded disturbances; and (iii) a combination of the two.

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The paper is organized as follows: In Section 2, we recall some standard notations and present the main results of the paper. In Section 3, we first recall the classical low-gain feedback design, which we use to develop a nonlinear dynamic low-gain feedback. In Section 4, we prove our main results based on the feedback laws developed in Section 3.

2. MAIN RESULT

We first recall some standard notations used in this paper. Let the vectors $e_1, \dots, e_n$ denote the standard basis for $\mathbb{R}^n$; that is, e_i is a unit vector with the ith entry equal to 1. For a vector $x \in \mathbb{R}^n$, $\|x\|$ denotes its Euclidean norm and x' denotes its transpose. For a matrix $X \in \mathbb{R}^{n\times m}$, $\|X\|$ denotes its induced 2-norm and X' denotes its transpose. For a positive-definite matrix $X \in \mathbb{R}^{n\times n}$, $C = X^{1/2} \in \mathbb{R}^{n\times n}$ is a non-singular matrix such that $X = C'C$.

We define a class of signal

$$\Omega_\infty = \left\{ d \in \mathcal{L}_\infty \mid \text{ there exsits } M > 0 \text{ such that } \left| \int_{t_1}^{t_2} d(t)\,\mathrm{d}t \right| \le M, \quad \forall\, t_2 \ge t_1 \ge 0 \right\}.$$

We now present the main theorems of this paper. The first theorem shows that if the disturbance is misaligned with the input (i.e., $B'E = 0$), then boundedness of the state can be ensured for any bounded disturbance by using a nonlinear static state feedback.

Theorem 1
Consider the system (1) with $B'E = 0$. There exists a nonlinear state feedback such that the closed-loop system satisfies the following properties:

1. In the absence of d, the origin is globally asymptotically stable.
2. If $d \in \mathcal{L}_\infty$, then $x \in \mathcal{L}_\infty$ for any $x(0) \in \mathbb{R}^n$.

If the disturbance is matched with the input (i.e. $E = B$), then the boundedness of the state trajectories can be preserved if the disturbance is integral-bounded.

Theorem 2
Consider the system (1) with $B = E$. There exists a nonlinear state feedback such that the closed-loop system satisfies the following properties:

1. In the absence of d, the origin is globally asymptotically stable.
2. If $d \in \Omega_\infty$, then $x \in \mathcal{L}_\infty$ for any $x(0) \in \mathbb{R}^n$.

We can combine the matched and the misaligned cases to obtain a more general case of mismatched disturbances. Specifically, consider the system

$$\dot{x} = Ax + B\sigma(u) + E_1 d_1 + E_2 d_2, \tag{3}$$

where $B'E_1 = 0$ and E_2 is given by

$$E_2 = \begin{bmatrix} \bar{E}_2 \\ \alpha \end{bmatrix}, \tag{4}$$

where α is either a non-zero real number or a row vector with only non-zero elements and $\bar{E}_2$ can be an arbitrary matrix with appropriate dimension. Based on the previous theorems, we can prove the following result.

Theorem 3
Consider the system (3). There exists a nonlinear state feedback such that the closed-loop system satisfies the following properties:

1. In the absence of d, the origin is globally asymptotically stable.
2. If $d_1 \in \mathcal{L}_\infty$ and $d_2 \in \Omega_\infty$, then $x \in \mathcal{L}_\infty$ for any $x(0) \in \mathbb{R}^n$.

3. CONTROLLER DESIGN

In this section we shall construct controllers that will be used to prove our main results. We start with a brief review of classical low-gain state feedback design.

3.1. Classical low-gain state feedback design

Classical low-gain feedback provides a family of parameterized stabilizing static feedback gains that vanish asymptotically as the parameter approaches zero. The philosophy behind classical low-gain design is that, by choosing the parameter small enough, the feedback gain can be made sufficiently small so that the saturation remains inactive in the whole state space or within any pre-specified compact subset. Classical low-gain design can be carried out using one of three approaches, namely, the method of direct eigen-structure assignment [7]; the H_2 and H_∞ ARE-based method [8, 18]; or the parametric Lyapunov-based method [21]. In this paper, we choose the parametric Lyapunov-based method because of its convenient properties when applied to a chain of integrators.

Consider the system (1) and let P_ε be the unique positive-definite solution of the parametric Riccati equation

$$A'P_\varepsilon + P_\varepsilon A - P_\varepsilon BB'P_\varepsilon + \varepsilon P_\varepsilon = 0. \tag{5}$$

The classical low-gain state feedback is given by

$$u = -B'P_\varepsilon x. \tag{6}$$

It is shown in [21] that (6) solves the semi-global stabilization problem for the system (1). In the global setting, the feedback takes the same form as in (6), but the low-gain parameter ε, instead of being fixed, is scheduled as a function of the state of the system. Such a scheduling has to satisfy the following properties for some design parameter $\delta \leq 1$.

1. There exists an open neighborhood O of the origin such that for all $x \in O$, $\varepsilon_a(x) = 1$.
2. For any $x \in \mathbb{R}^n$, $|B'P_{\varepsilon_a(x)}x| \leq \delta$.
3. $\varepsilon_a(x) \to 0 \implies \|x\| \to \infty$.
4. For each $c > 0$, the set $\{x \in \mathbb{R}^n \mid x'P_{\varepsilon_a(x)}x \leq c\}$ is bounded.
5. There is a function $g\colon [0,\infty) \to (0,1]$ such that for all $x \neq 0$, $\varepsilon_a(x) = g(x'P_{\varepsilon_a(x)}x)$.

A particular choice of $\varepsilon_a(x)$, given in [9], is

$$\varepsilon_a(x) = \max\left\{r \in (0,1] \mid (x'P_r x) \times (B'P_r B) \leq \delta^2\right\}, \tag{7}$$

where P_r is the solution of (5) with $\varepsilon = r$. Based on this scheduling, the feedback law is given by

$$u = -B'P_{\varepsilon_a(x)}x. \tag{8}$$

3.2. Dynamic low-gain state feedback design

We now consider the chain of integrators and construct controllers that will be used to prove all the theorems of Section 2. For the case of misaligned disturbances, which is treated in Theorem 1, we can simply apply the classical scheduled low-gain state feedback (8) and (7) with $\delta = 1$. However, for the matched case, treated in Theorem 2, and the combined case, treated in Theorem 3, we construct a dynamic controller as follows:

$$\begin{cases} \dot{y} = \sigma(-B'P_{\varepsilon_a(\bar{x})}\bar{x}), \\ u = -B'P_{\varepsilon_a(\bar{x})}x, \end{cases} \tag{9}$$

where $P_{\varepsilon_a(\bar{x})}$ is the solution of (5) with

$$\varepsilon = \varepsilon_a(\bar{x}) := \max\left\{r \in [0,1] \mid (\bar{x}'P_r\bar{x}) \times (B'P_rB) \leq \tfrac{1}{4}\right\} \tag{10}$$

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and

$$\bar{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \\ y \end{bmatrix}.$$

4. PROOFS OF MAIN RESULTS

We first prove Theorem 1 for the misaligned case $B'E = 0$.

Proof of Theorem 1
Consider the scheduled static low-gain state feedback (8) and (7) with $\delta = 1$.

Define a Lyapunov function $V(x) = x'P_{\varepsilon_a(x)}x$. Differentiating $V(x)$ along the trajectories yields

$$\begin{aligned}\dot{V} &= x'A'P_{\varepsilon_a(x)}x + x'P_{\varepsilon_a(x)}Ax - 2x'P_{\varepsilon_a(x)}BB'P_{\varepsilon_a(x)}x + 2x'P_{\varepsilon_a(x)}Ed + x'\tfrac{\mathrm{d}P_{\varepsilon_a(x)}}{\mathrm{d}t}x \\ &\leq -\varepsilon V + 2x'P_{\varepsilon_a(x)}Ed + x'\tfrac{\mathrm{d}P_{\varepsilon_a(x)}}{\mathrm{d}t}x.\end{aligned}$$

In absence of d, we have that

$$\dot{V} \leq -\varepsilon V + x'\tfrac{\mathrm{d}P_{\varepsilon_a(x)}}{\mathrm{d}t}x$$

It was shown in [6] that (7) implies that $\dot{V}$ and $x'\frac{\mathrm{d}P_{\varepsilon_a(x)}}{\mathrm{d}t}x$ can not have the same sign. Therefore, we find that

$$\dot{V} < 0$$

for all $x \in \mathbb{R}^n$. This shows global asymptotic stability. We proceed to prove Property 2. Lemma 4 given in the appendix implies that if $B'E = 0$, then there exists an $M > 0$ depending on system data such that

$$\|P_\varepsilon^{1/2}E\| \leq \varepsilon M$$

for $\varepsilon \in [0, 1]$.

For $d \in \mathcal{L}_\infty$, we have

$$\begin{aligned}\dot{V} &\leq -\varepsilon V + 2\|x'P_{\varepsilon_a(x)}^{1/2}\|\|P_{\varepsilon_a(x)}^{1/2}E\|\|d\|_\infty + x'\tfrac{\mathrm{d}P_{\varepsilon_a(x)}}{\mathrm{d}t}x \\ &\leq -\varepsilon V + 2\varepsilon M\sqrt{V}\|d\|_\infty + x'\tfrac{\mathrm{d}P_{\varepsilon_a(x)}}{\mathrm{d}t}x \\ &= -\varepsilon\sqrt{V}(\sqrt{V} - 2M\|d\|_\infty) + x'\tfrac{\mathrm{d}P_{\varepsilon_a(x)}}{\mathrm{d}t}x.\end{aligned}$$

For $V \geq 4M^2\|d\|_\infty^2$, we have

$$\dot{V} \leq x'\tfrac{\mathrm{d}P_{\varepsilon_a(x)}}{\mathrm{d}t}x.$$

The scheduling (7) guarantees that $\dot{V}$ and $x'\frac{\mathrm{d}P_{\varepsilon_a(x)}}{\mathrm{d}t}x$ cannot have the same sign. This implies that $\dot{V} < 0$ for $V \geq 4M^2\|d\|_\infty^2$. Hence, V is bounded for all $t \geq 0$. Boundedness of x follows from Property 4 of the scheduling. □

Next, we proceed to the matched case $E = B$.

Proof of Theorem 2
Consider the nonlinear dynamic low-gain state feedback controller (9) and (10). Define $\bar{y} = x_n - y$. We have

$$\dot{\bar{y}} = \sigma(-B'P_{\varepsilon_a(\bar{x})}x) - \sigma(-B'P_{\varepsilon_a(\bar{x})}\bar{x}) + d.$$

Note that $\bar{x} = x - Bx_n + By = x - B\bar{y}$. Hence

$$\dot{\bar{y}} = \sigma(-B'P_{\varepsilon_a(\bar{x})}\bar{x} - B'P_{\varepsilon_a(\bar{x})}B\bar{y}) - \sigma(-B'P_{\varepsilon_a(\bar{x})}\bar{x}) + d.$$

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We therefore have

$$\dot{\bar{x}} = \dot{x} - B\dot{\bar{y}} = A\bar{x} + B\sigma(-B'P_{\varepsilon_a(\bar{x})}\bar{x}) + e_{n-1}\bar{y}.$$

In the new coordinates $(\bar{x}, \bar{y})$, the closed-loop system is given by

$$\begin{cases} \dot{\bar{x}} = A\bar{x} + B\sigma(-B'P_{\varepsilon_a(\bar{x})}\bar{x}) + e_{n-1}\bar{y}, \\ \dot{\bar{y}} = \sigma(-B'P_{\varepsilon_a(\bar{x})}\bar{x} - B'P_{\varepsilon_a(\bar{x})}B\bar{y}) - \sigma(-B'P_{\varepsilon_a(\bar{x})}\bar{x}) + d. \end{cases} \tag{11}$$

We first show global asymptotic stability in the absence of disturbances. Close to the origin, we have $\varepsilon_a(\bar{x}) = 1$, and all the saturations are inactive. Equation (11) then reduces to a linear system

$$\begin{cases} \dot{\bar{x}} = A\bar{x} - BB'P_1\bar{x} + e_{n-1}\bar{y}, \\ \dot{\bar{y}} = -B'P_1B\bar{y}, \end{cases}$$

where P_1 is the solution of (5) with $\varepsilon = 1$. Local stability is therefore obvious. To prove global attractivity, consider the dynamics of $\bar{y}$. Define a Lyapunov function $V_1 = \bar{y}^2$. We then have

$$\dot{V}_1 = 2\bar{y}\left[\sigma(-B'P_{\varepsilon_a(\bar{x})}\bar{x} - B'P_{\varepsilon_a(\bar{x})}B\bar{y}) - \sigma(-B'P_{\varepsilon_a(\bar{x})}\bar{x})\right].$$

The scheduling (10) guarantees that $|B'P_{\varepsilon_a(\bar{x})}\bar{x}| \leq \frac{1}{2}$. Therefore, owing to Lemma 6 in the appendix, we find that

$$\dot{V}_1 \leq -\bar{y}\sigma(B'P_{\varepsilon_a(\bar{x})}B\bar{y}). \tag{12}$$

This shows that $\bar{y}$ is bounded. Since $B'e_{n-1} = 0$, Theorem 1 implies that $\bar{x}$ is bounded for all $t \geq 0$. Hence $\varepsilon_a(x)$ is bounded away from zero, which, together with (12) implies $\bar{y} \to 0$ as $t \to 0$.

Next consider the dynamics of $\bar{x}$. Define another Lyapunov function $V_2(\bar{x}) = \bar{x}'P_{\varepsilon_a(\bar{x})}\bar{x}$ and a set

$$\mathcal{K} = \{\bar{x} \mid V_2(\bar{x}) \leq \tfrac{1}{2B'P_1B}\}.$$

It can be easily seen from (10) that for $\bar{x} \in \mathcal{K}$, $\varepsilon_a(\bar{x}) = 1$. Differentiating V_2 along the trajectory, we have

$$\begin{aligned} \dot{V}_2 &\leq -\varepsilon_a(\bar{x})V_2 + 2\bar{x}'P_{\varepsilon_a(x)}e_{n-1}\bar{y} + \bar{x}'\tfrac{\mathrm{d}P_{\varepsilon_a(\bar{x})}}{\mathrm{d}t}\bar{x} \\ &\leq -\varepsilon_a(\bar{x})V_2 + 2|\bar{y}|\sqrt{V_2}\|P^{1/2}_{\varepsilon_a(\bar{x})}e_{n-1}\| + \bar{x}'\tfrac{\mathrm{d}P_{\varepsilon_a(\bar{x})}}{\mathrm{d}t}\bar{x} \\ &\leq -\varepsilon_a(\bar{x})V_2 + 2M_2\varepsilon_a(\bar{x})|\bar{y}|\sqrt{V_2} + \bar{x}'\tfrac{\mathrm{d}P_{\varepsilon_a(\bar{x})}}{\mathrm{d}t}\bar{x} \\ &= -\varepsilon_a(\bar{x})\sqrt{V_2}(\sqrt{V_2} - 2M_2|\bar{y}|) + \bar{x}'\tfrac{\mathrm{d}P_{\varepsilon_a(\bar{x})}}{\mathrm{d}t}\bar{x}. \end{aligned}$$

Since $\bar{y} \to 0$, for given $\bar{y}(0)$ and $\bar{x}(0)$, there exists a T such that $|y(t)| \leq \min\{\frac{1}{2}, \frac{1}{4M_2\sqrt{B'P_1B}}\}$ for $t \geq T$. Therefore, for $t \geq T$ and $\bar{x} \notin \mathcal{K}$, $\sqrt{V_2} - 2M_2|\bar{y}| \geq \frac{\sqrt{V_2}}{2}$, and thus

$$\dot{V}_2 \leq -\tfrac{\varepsilon_a(\bar{x})}{2}V_2 + \bar{x}'\tfrac{\mathrm{d}P_{\varepsilon_a(\bar{x})}}{\mathrm{d}t}\bar{x}.$$

Since $\dot{V}_2$ cannot have the same sign as $\bar{x}'\frac{\mathrm{d}P_{\varepsilon_a(\bar{x})}}{\mathrm{d}t}\bar{x}$, we conclude that $\dot{V}_2 < 0$ for $\bar{x} \notin \mathcal{K}$. This implies that $\bar{x}$ will enter $\mathcal{K}$ within finite time after $t = T$ and remain in $\mathcal{K}$ thereafter. For $t > T$ and $\bar{x} \in \mathcal{K}$, we have $\varepsilon_a(\bar{x}) = 1$ and $|y| \leq \frac{1}{2}$. All saturations are inactive and the system becomes linear. It therefore follows that $\bar{x} \to 0$, which shows that the origin is globally attractive.

When disturbances are present, Lemma 5 shows that $|\bar{y}| \in \mathcal{L}_\infty$ given $d \in \Omega_\infty$. Boundedness of $\bar{x}$ therefore follows from Theorem 1. □

Finally, we prove Theorem 3 for the combined case by using Theorems 1 and 2.

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Proof of Theorem 3
This proof is basically a combination of those of Theorems 1 and 2. Consider the dynamic low-gain state feedback (9) and the scheduling (10). Define $\bar{y} = x_n - y$. We have

$$\dot{\bar{y}} = \sigma(-B'P_{\varepsilon_a(\bar{x})}x) - \sigma(-B'P_{\varepsilon_a(\bar{x})}\bar{x}) + \alpha d_2.$$

Note that $\bar{x} = x - Bx_n + By = x - B\bar{y}$. Hence

$$\dot{\bar{y}} = \sigma(-B'P_{\varepsilon_a(\bar{x})}x) - \sigma(-B'P_{\varepsilon_a(\bar{x})}x + B'P_{\varepsilon_a(\bar{x})}B\bar{y}) + \alpha d_2.$$

Lemma 5 shows that $|\bar{y}| \in \mathcal{L}_\infty$ given $d \in \Omega_\infty$ for any $\bar{y}(0)$. We have

$$\dot{\bar{x}} = \dot{x} - B\dot{\bar{y}} = A\bar{x} + B\sigma(-B'P_{\varepsilon_a(\bar{x})}\bar{x}) + \begin{bmatrix} E_1 & \tilde{E}_2 & e_{n-1} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ \bar{y} \end{bmatrix},$$

where

$$\tilde{E}_2 = \begin{bmatrix} \bar{E}_2 \\ 0 \end{bmatrix}.$$

Note that

$$B' \begin{bmatrix} E_1 & \tilde{E}_2 & e_{n-1} \end{bmatrix} = 0.$$

The rest of the proof now proceeds in the same way as the proof of Theorem 1. □

5. EXAMPLE

We conclude the paper with an example. Consider the following system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \sigma(u) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} d_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} d_2$$

where $d_1 = \sin t$ and $d_2 = 2$. The controller u is given by (9) and (10). The simulation data is shown in the following figure:

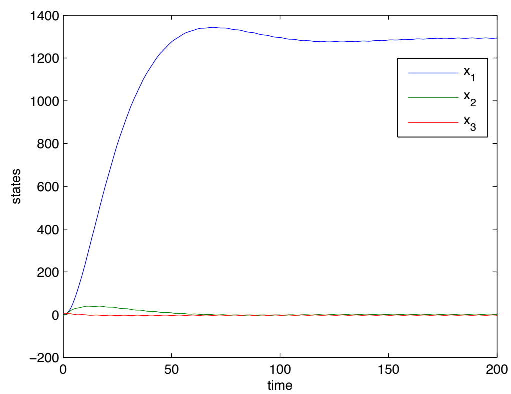

Figure 1. Triple integrator with actuator saturation and disturbances

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APPENDIX

The following lemma regarding the properties of P_ε is adapted from [21].

Lemma 4
The parametric Riccati equation (5) associated with data A, B given by (2) has a unique positive-definite solution P_ε with the following properties:

1. P_ε is a polynomial matrix in ε.
2. $P_\varepsilon \to 0$ as $\varepsilon \to 0$.
3. $\frac{\mathrm{d}P_\varepsilon}{\mathrm{d}\varepsilon} > 0$ for all $\varepsilon \in [1, 0)$.
4. There exists an $M > 0$ such that for any $\varepsilon \in [0, 1]$,

$$e_i' P_\varepsilon e_i \leq M\varepsilon^2,$$

where $i < n$ and e_i is a unit vector whose ith entry is 1.

Proof
The first three properties were proven in [21]. Regarding Property 4, it was shown in [21] (see Lemma 1) that the unique positive-definite solution $P_\varepsilon = [p_{i,j}]_{n\times n}$ to the parametric Riccati equation associated with A, B given by (2) can be computed using the following recursion: for $i = n-1, \ldots, 0$

$$p_{i+1,n} = p_{n,i+1} = (-1)^{i+1}\left[\sum_{i+1}^{n}(-1)^k p_{n,k+1} C_k^{k-i}\varepsilon^{k-i} + (-1)^n C_n^{n-i}\varepsilon^{n-i}\right]$$

with $C_k^{k-i} = \frac{k!}{i!(k-i)!}$ and $p_{n,n+1} = 0$. For $k = j, j-1, \ldots, 1$ and $j = n-1, \ldots, 1$

$$p_{k,j} = p_{k,n}p_{n,j+1} - p_{j+1,k-1} - \varepsilon p_{k,j+1}$$

with $p_{i,0} = p_{0,i} = 0$.

This shows that P_ε is a polynomial matrix in ε and for $i < n$ and $j < n$, $p_{i,j}$ is at least of order ε^2. Therefore, for $i < n$, $e_i' P_\varepsilon e_i$ is at least of order ε^2. □

Lemma 5
Consider the system

$$\dot{y}(t) = \sigma(v(t)) - \sigma(v(t) + k(t)y) + d, \tag{13}$$

where $d \in \Omega_\infty$ and $k(t) > 0$ and $v(t)$ are continuous. We have then $y \in \mathcal{L}_\infty$ for all $y(0)$.

Proof
Define

$$\dot{\bar{y}} = d, \quad \bar{y}(0) = y(0).$$

Since $d \in \Omega_\infty$, there exists a $M > 0$ such that $|\bar{y}(t)| \leq |y(0)| + M$ for all $t > 0$. Define $\tilde{y} = y - \bar{y}$. We have

$$\dot{\tilde{y}} = \sigma(v) - \sigma(v + k(\tilde{y} + \bar{y})), \quad \tilde{y} = 0.$$

Let $\tilde{V} = \tilde{y}^2$. Taking the derivative of $\tilde{V}$ with respect to t, we get

$$\dot{\tilde{V}} = \tilde{y}\left[\sigma(v) - \sigma(v + k(\tilde{y} + \bar{y}))\right].$$

If $\tilde{V} \geq (|y(0)| + M)^2$, then $|\tilde{y}| \geq M + |y(0)| \geq |\bar{y}|$. But this implies that $k(\tilde{y} + \bar{y})$ has the same sign as $\tilde{y}$. Thus

$$\dot{\tilde{V}} = \tilde{y}\left[\sigma(v) - \sigma(v + k(\tilde{y} + \bar{y}))\right] \leq 0.$$

Since $\tilde{V}(0) = 0$, we have $\tilde{V} \leq (|y(0)| + M)^2$ and $|\tilde{y}| \leq |y(0)| + M$ for all $t > 0$. Therefore, $|y| \leq |\bar{y}| + |\tilde{y}| \leq 2M + 2y(0)$. □

The following lemma was shown in [12]:

Lemma 6
For any $w \in \mathbb{R}^m$ satisfying $\|w\| \leq \frac{1}{2}$ we have

$$2u'[\sigma(w) - \sigma(w-u)] \geq u'\sigma(u)$$

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