Control of linear systems with input saturation and non-input-additive sustained disturbances – Discrete-time systems

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Abstract— **We study in this paper the control of discrete-time neutrally stable and critically unstable linear systems subject to input saturation and disturbances. It is shown that a feedback controller can be designed so that the closed-loop states remain bounded for two classes of disturbances and the equilibrium is globally asymptotically stable. This paper is a companion of [7].**

I. Introduction

In this paper, we study the control of discrete-time linear systems subject to input saturation and external disturbances. The paper is a companion of [7], in which we show that for continuous-time critically unstable or neutrally stable systems with input saturation, bounded closed-loop states in the presence of disturbances, as well as a globally asymptotically stable equilibrium in the absence of disturbances, can be achieved by a properly designed feedback controller if

1) disturbances are *aligned* and belong to Ω_∞ set which includes all $\mathcal{L}_\infty$ signals that do not contain large frequency component corresponding to system eigenvalues on the imaginary axis, or
2) disturbances are *misaligned* and bounded, i.e. belong to $\mathcal{L}_\infty$.

In this paper, we shall prove a discrete-time counterpart of the above result.

A summary of research in the context of external stability of continuous-time linear systems subject to input saturation is given in [7]. It is prudent to make a complementary review of the literature with respect to previous studies for discrete-time systems. For input-additive disturbances, a nonlinear low-and-high gain state feedback is designed in [10] to achieve simultaneous internal and ℓ_p stabilization with finite gain. In the special case of open-loop neutrally stable systems, it has been shown that a linear state feedback achieves ℓ_p stability with finite gain for $p \in [1,\infty]$ while rendering the origin globally asymptotically stable without disturbances [1], [2]. On the other hand, ℓ_p stabilization with finite gain has been shown to be impossible in the non-input-additive case, but ℓ_p stabilization without finite gain is always attainable via a dynamic low-gain feedback [5]. Moreover, for an open-loop neutrally stable system, it is attainable via a linear state feedback [4]. Nevertheless, these results are limited to ℓ_p disturbances for $p \in [1,\infty)$ (i.e., disturbances that vanish asymptotically), and not to sustained signals belonging to ℓ_∞.

Of particular interest in dealing with sustained signals that are non-additive, is the research on identifying classes of ℓ_∞ disturbances for which a controller can be designed to yield bounded closed-loop state trajectories. Work along this line has been carried out by our group for continuous-time systems in [11], [9], [8] and most recently in [7]. In this paper, we shall show that despite various differences between continuous- and discrete-time systems, the underlying idea behind this body of works carries over to its discrete-time counterpart.

Like in [7], we first establish the result for discrete-time neutrally stable systems having semi-simple eigenvalues on the imaginary axis. Roughly speaking, we shall show that for disturbances that do not have large sustained frequency components corresponding to the system's eigenvalues on the unit circle, a linear static state feedback can be employed to achieve boundedness of the trajectories for any initial condition and render the origin global asymptotically stable.

Furthermore, based on the construction for neutrally stable systems, we shall extend results to general critically unstable systems which may have degenerate eigenvalues on the unit circle. It will be shown that two classes of disturbances can be coped with by a properly designed feedback controller. At the same time, the resulting closed-loop system is globally asymptotically stable.

The paper is organized as following: Preliminaries are given in Section II. Results for neutrally stable systems and critically unstable systems are respectively presented in Section III and IV. Some technical results used in the proof are given in the Appendix.

II. Preliminaries

Consider the following system

$$x^+ = Ax + B\sigma(u) + Ed, \quad x(0) = x_0, \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $d \in \mathbb{R}^p$. In this paper, we are concerned with sustained disturbances, for which we assume that $d \in \ell_\infty$. We also use $\ell_\infty(\delta)$ to denote the set of ℓ_∞ signals whose ℓ_∞ norm is less than δ. We also assume that (A, B) is stabilizable and A has all its eigenvalues in the closed unit disc.

Moreover, the system is said to be *neutrally stable* if

1) A has all its eigenvalues in the closed unit disc;
2) A has at least one eigenvalue on the unit circle;

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3) those eigenvalues on the unit circle are semi-simple.

The system is said to be *critically unstable* if

1) A has all its eigenvalues in the closed unit disc;
2) A has at least one eigenvalue on the unit circle which has Jordan block size greater than 1.

A system that is neutrally stable or critically unstable can be decomposed into the following form:

$$\begin{bmatrix} x_s^+ \\ x_u^+ \end{bmatrix} = \begin{bmatrix} A_s & \\ & A_u \end{bmatrix} \begin{bmatrix} x_s \\ x_u \end{bmatrix} + \begin{bmatrix} B_s \\ B_u \end{bmatrix} \sigma(u) + \begin{bmatrix} E_s \\ E_u \end{bmatrix} d,$$

where A_s is Schur stable, (A_u, B_u) is controllable and A_u has all its eigenvalues on the unit circle. Since A_s is Schur stable and $\sigma(u)$ and d are bounded, it follows that the x_s dynamics will remain bounded regardless of controllers. Therefore, without loss of generality, we can ignore the stable dynamics and assume in (1) that (A, B) is controllable and all the eigenvalues of A are on the unit circle.

A. Extended class of disturbances

We define a set of discrete disturbances

$$\Omega_\infty = \Bigg\{ d \in \ell_\infty \mid \exists M > 0, \ \ s.\ t.\ \forall i \in 1, \dots, \ell, \\ \forall k_2 \geq k_1 \geq 0, \quad \left\| \sum_{k=k_1}^{k_2} d(k) z_i^k \right\| \leq M \Bigg\}, \quad (2)$$

where $z_i = e^{j\theta_i}$, $i = 1, ..., \ell$, denotes the eigenvalues of A. Definition (2) implies all partial sums of the power series are bounded at $z = z_i$. The set Ω_∞ contains signals which do not have sustained component at discrete frequency θ_i.

III. Neutrally stable systems

In this section, we deal with discrete-time neutrally stable systems. We assume that (A, B) is controllable and $A'A = I$.

We use a linear state feedback controller $u = -\kappa B'Ax$ which gives a closed-loop system as

$$x^+ = Ax(k) + B\sigma(-\kappa B'Ax) + Ed, \quad x(0) = x_0.$$

For sufficiently small κ, global asymptotic stability follows from Lemma 2 in [6]. As such, we focus here only on the boundedness of closed-loop states in the presence of disturbances.

A. Single-frequency systems

We start by considering an example system with a pair of complex eigenvalues at $\pm j$:

$$\begin{bmatrix} x_1^+ \\ x_2^+ \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sigma(\kappa x_1) + \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} d, \qquad (3)$$

where $d \in \Omega_\infty$.

Theorem 1: For $\kappa \leq \frac{1}{4}$, the trajectories of (3) remain bounded for any initial condition.

Proof: To analyze the system, we introduce a rotation matrix

$$R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^k,$$

which represents a counterclockwise rotation by an angle $\frac{k}{2}\pi$. The dynamics of the rotation matrix are given by

$$R^+ = R \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

We can study the dynamics of x from a rotated coordinate frame, and toward this end we define the rotated state $y = Rx$. The dynamics of y is given by

$$y^+ = y + R^+ \left[\begin{bmatrix} 0 \\ 1 \end{bmatrix} \sigma\left(\begin{bmatrix} \kappa & 0 \end{bmatrix} R'y \right) + \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} d \right],$$

with $y(0) = x(0) = x_0$. Next, define a fictitious system

$$\tilde{y}^+ = \tilde{y} + R^+ \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} d, \quad \tilde{y}(0) = x_0. \qquad (4)$$

The solution to this dynamic system is

$$\tilde{y}(k) = \tilde{y}(0) + \sum_{j=0}^{k-1} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{j+1} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} d(j)$$

It follows from the definition of Ω_∞ that the sum in the right-hand side is bounded for all k, and hence $\tilde{y} \in \ell_\infty$.

Consider the difference between y and the fictitious state $\tilde{y}$, given by $z = y - \tilde{y}$, with dynamics

$$z^+ = z + R^+ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sigma\left(\begin{bmatrix} \kappa & 0 \end{bmatrix} R'z + \delta \right),$$

with $z(0) = 0$ where $\delta = (\kappa, 0)R'\tilde{y} \in \ell_\infty$. We rotate z back to the original coordinate frame by introducing $w = R'z$, thereby obtaining the dynamics

$$w^+ = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} w + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sigma(\begin{bmatrix} \kappa & 0 \end{bmatrix} w + \delta), \quad w(0) = 0.$$

It is shown in Lemma 3 that for $\kappa \leq 1/4$, the above system is ℓ_∞ stable with respect to the input δ for sufficiently small κ, and hence $w \in \ell_\infty$. Finally, we have that $x = w + R'\tilde{y}$, and hence $x \in \ell_\infty$. ■

To demonstrate the importance of the disturbance belonging to Ω_∞, we shall now show that if d contains a large component at discrete frequency $\pm\frac{\pi}{2}$, the states of (3) will diverge toward infinity for any initial condition. Suppose that $d(k) = a\sin(k\pi/2 + \theta)$, where a is an amplitude yet to be chosen. For ease of presentation, we assume that $[e_1, e_2]' = [0, 1]'$. Consider the dynamics of the rotated state y from the proof of Theorem 1. We have

$$\begin{aligned} y^+ &= y + R^+ \left[\begin{bmatrix} 0 \\ 1 \end{bmatrix} \sigma(s) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} a\sin(k\pi/2 + \theta) \right], \\ &= y + a \begin{bmatrix} -\cos(\frac{k}{2}\pi)\sin(\frac{k}{2}\pi + \theta) \\ -\sin(\frac{k}{2}\pi)\sin(\frac{k}{2}\pi + \theta) \end{bmatrix} - \begin{bmatrix} \cos(\frac{k}{2}\pi) \\ \sin(\frac{k}{2}\pi) \end{bmatrix} \sigma(s). \end{aligned}$$

where $s = (\kappa \ \ 0)R'y$. Using appropriate trigonometric identities, the dynamics can be rewritten as

$$y^+ = y - \frac{a}{2} \begin{bmatrix} \sin(k\pi + \theta) + \sin(\theta) \\ \cos(\theta) - \cos(k\pi + \theta) \end{bmatrix} - \begin{bmatrix} \cos(\frac{k}{2}\pi) \\ \sin(\frac{k}{2}\pi) \end{bmatrix} \sigma(s).$$

We have that either $|\sin(\theta)| \geq \sqrt{2}/2$ or $|\cos(\theta)| \geq \sqrt{2}/2$. Without loss of generality, we assume that $|\cos(\theta)| \geq \sqrt{2}/2$.

Let a be chosen such that $a \geq 4/\sqrt{2}(1+\varepsilon)$, where ε is a positive number. For the trajectory $y_2(k)$, we have

$$
\begin{aligned}
|y_2(k)| = \Big| y_2(0) - \sum_{i=0}^{k-1} \frac{a}{2}\Big(\cos(\theta) - \cos(i\pi + \theta)\Big) \\
- \sum_{i=0}^{k-1} \sin(\frac{i}{2}\pi)\sigma(s) \Big|.
\end{aligned}
$$

Noting that $\sin(i\pi/2)\sigma(\cdot)$ is bounded by ± 1, and using the bound $|a/2\cos(\theta)| \geq \sqrt{2}a/4 \geq 1+\varepsilon$, we therefore have

$$
\begin{aligned}
|y_2(k)| &\geq -|y_2(0)| - \frac{a}{2}\left|\sum_{i=0}^{k-1}\cos(i\pi+\theta)\right| + \sum_{i=0}^{k-1}\varepsilon \\
&\geq -|y_2(0)| - \frac{a}{2} + \varepsilon k.
\end{aligned}
$$

This shows that $y_2(k)$ diverges toward infinity.

B. Multi-frequency systems

We shall extend the results for single frequency systems to general neutrally stable system which may have different eigen-frequencies.

Theorem 2: Consider the system

$$
x^+ = Ax - B\sigma(\kappa B'Ax) + Ed, \quad x(0) = x_0 \tag{5}
$$

where (A, B) is controllable, $A'A = I$ and $d \in \Omega_\infty$. For κ such that $4\kappa B'B \leq I$, we have $x(k)$ bounded for all $k \geq 0$ and for any initial condition.

The next theorem shows that a small disturbance that does not belong to Ω_∞ can also be tolerated.

Theorem 3: Consider the discrete-time system

$$
x^+ = Ax - B\sigma(\kappa B'Ax) + E_1 d_1 + E_2 d_2, \quad x(0) = x_0 \tag{6}
$$

where $d_1 \in \Omega_\infty$ and $d_2 \in \ell_\infty(\delta)$ with δ sufficiently small. Then for κ such that $4\kappa B'B \leq I$, we have $x(k)$ bounded for all $k \geq 0$ and for any initial condition.

The proofs of Theorem 2 and 3 can be found in [6].

IV. Critically unstable systems

A. Formulation

Consider a linear system with input saturation and disturbances:

$$
x^+ = Ax + B\sigma(u) + Ed, \quad x(0) = x_0 \tag{7}
$$

where (A, B) is controllable and A has all its eigenvalues on the unit circle. Suppose the eigenvalues of A have q different Jordon block sizes denoted by $n_1, ..., n_q$. Without loss of generality, we can assume $x = (x_1', x_2', \ldots, x_q')'$, and A, B are in the following form

$$
A = \begin{bmatrix} \bar{A}_1 & 0 & \cdots & 0 \\ 0 & \bar{A}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & \bar{A}_q \end{bmatrix}, B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_q \end{bmatrix}, \; E = \begin{bmatrix} E_1 \\ E_2 \\ \vdots \\ E_q \end{bmatrix} \tag{8}
$$

where

$$
x_i = \begin{bmatrix} x_{i,1} \\ x_{i,2} \\ \vdots \\ x_{i,n_i-1} \\ x_{i,n_i} \end{bmatrix}, \quad \bar{A}_i = \underbrace{\begin{bmatrix} A_i & I & 0 & \ldots & 0 \\ 0 & A_i & I & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & A_i & I \\ 0 & \cdots & \cdots & 0 & A_i \end{bmatrix}}_{n_i \times n_i \text{ blocks}},
$$

$$
B_i = \begin{bmatrix} B_{i,1} \\ B_{i,2} \\ \vdots \\ B_{i,n_i-1} \\ B_{i,n_i} \end{bmatrix}, \quad E_i = \begin{bmatrix} E_{i,1} \\ E_{i,2} \\ \vdots \\ E_{i,n_i-1} \\ E_{i,n_i} \end{bmatrix}, \tag{9}
$$

$x_{i,j} \in \mathbb{R}^{p_i}$ with $n = \sum_{i=1}^{q} n_i p_i$ and $A_i' A_i = I$. Note that the above form can be obtained by assembling together in the real Jordan canonical form those blocks corresponding to eigenvalues with the same Jordan block size.

We say the disturbance d is *aligned* if $E_{i,n_i} \neq 0$ for some $i = 1, \ldots, q$ and *misaligned* if $E_{i,n_i} = 0$ for all $i = 1, \ldots, q$.

Without loss of generality, for any critically unstable system with input saturation and non-input-additive disturbances as given by (7), (8) and (9), we can equivalently rewrite the system in the following form

$$
\rho x = Ax + B\sigma(u) + \bar{E}_1 d_1 + \bar{E}_2 d_2 + \bar{E}_3 d_3, \tag{10}
$$

with $x(0) = x_0$. In the above system, d_1 is misaligned and contain arbitrary disturbances that belong to $\mathcal{L}_\infty$ (continuous-time) or ℓ_∞ (discrete-time), d_2 contains all aligned disturbances belonging to Ω_∞ and d_3 contains aligned disturbances which do not belong to Ω_∞ but are sufficiently small. The system data A and B are given by (8) and (9). The $\bar{E}_1$, $\bar{E}_2$ and $\bar{E}_3$ are in the form

$$
\bar{E}_1 = \begin{bmatrix} \bar{E}_{1,1} \\ \vdots \\ \bar{E}_{1,q-1} \\ \bar{E}_{1,q} \end{bmatrix}, \; \bar{E}_{1,i} = \begin{bmatrix} E_{i,1} \\ \vdots \\ E_{i,n_i-1} \\ 0 \end{bmatrix} \tag{11}
$$

and

$$
\bar{E}_j = \begin{bmatrix} \bar{E}_{j,1} \\ \vdots \\ \bar{E}_{j,q-1} \\ \bar{E}_{j,q} \end{bmatrix}, \; \bar{E}_{j,i} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ E_{i,n_i}^{j} \end{bmatrix}, \quad j = 2, 3. \tag{12}
$$

B. Controller design

We shall next design a nonlinear dynamic state feedback controller which will be able to solve our problem. Let (A, B) satisfy the assumptions made in the preceding section and $P(\varepsilon) > 0$ be the solution to a Discrete Parametric Lyapunov Equation (DPLE)

$$
\begin{aligned}
(1-\varepsilon)P(\varepsilon) = A'P(\varepsilon)A \\
- A'P(\varepsilon)B(B'P(\varepsilon)B + I)^{-1}B'P(\varepsilon)A.
\end{aligned} \tag{13}
$$

The existence of $P(\varepsilon)$ for $\varepsilon \in (0,1)$ has been established in [12]. When A is given by (8) and (9), an important property of $P(\varepsilon)$ is shown in the following lemma. The proof can be found in Appendix.

Lemma 1: Let $P(\varepsilon)$ be the solution to DPLE (13) associated with A and B given by (8) and (9). For any matrix $\bar{E}_1$ in the form of (11), there exists M such that for $\varepsilon \in (0,1]$

$$\bar{E}_1' P(\varepsilon)\bar{E}_1 \leq M\varepsilon^2 I$$

Consider the following dynamic state feedback controller

$$\begin{cases} \hat{x}_i^+ = A_i\hat{x}_i + B_{i,n_i}\sigma\left(-F(\varepsilon)\bar{x}\right), \quad i = 1,\dots,q \\ u = -\kappa\hat{B}'\hat{A}(x_b - \hat{x}) - F(\varepsilon)\bar{x}, \end{cases} \tag{14}$$

where κ is such that $8\kappa B'B \leq I$, $\hat{x} = [\hat{x}_1', \hat{x}_2,' \cdots, \hat{x}_q']'$ and

$$x_b = \begin{bmatrix} x_{1,n_1} \\ x_{2,n_2} \\ \vdots \\ x_{q,n_q} \end{bmatrix}, \ \bar{x} = \begin{bmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_{q-1} \\ \bar{x}_q \end{bmatrix}, \ \bar{x}_i = \begin{bmatrix} x_{i,1} \\ \vdots \\ x_{i,n_i-1} \\ \hat{x}_i \end{bmatrix}.$$

and

$$\hat{A} = \begin{bmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_q \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} B_{1,n_1} \\ B_{2,n_2} \\ \vdots \\ B_{q,n_q} \end{bmatrix},$$
$$F(\varepsilon) = (B'P(\varepsilon)B + I)^{-1}B'P(\varepsilon)A,$$

$P(\varepsilon)$ is the solution to DPLE (13) and ε is determined by

$$\varepsilon = \varepsilon_a(\bar{x}) := \max\{r \in (0, 0.9] \mid (\bar{x}'P(r)\bar{x}) \operatorname{trace}(P_r) \leq \tfrac{\delta^2}{b}\} \tag{15}$$

where $b = 2\operatorname{trace}(BB')$, $P(r)$ is the unique positive definite solution of DPLE (13) with $\varepsilon = r$, $\bar{x}$ is x with x_{i,n_i} replaced by $\hat{x}_i$. The above scheduling of ε guarantees that $\|F(\varepsilon_a(\bar{x}))\bar{x}\| \leq \frac{1}{4}$ for any $\bar{x}$.

Note that $\bar{x}$ is the system state x with bottom state segment x_{i,n_i} of each Jordan block $\bar{A}_i$ replaced by controller states $\hat{x}_i$. The feedback input is generated based on $\bar{x}$ instead of x. As will become clear in the proof, the underlying idea behind (14) is that by utilizing the states of controller and the property of Ω_∞, we will be able to convert some aligned disturbances affecting the bottom states into misaligned disturbances which turns out to be less restricted.

C. Main results

Theorem 4: Consider the system (10) with controller (14). For κ such that $8\kappa B'B \leq I$, we have that

1) in the absence of d_1, d_2 and d_3, the origin is globally asymptotically stable;
2) there exists a $\delta_1 > 0$ such that the state remains bounded for any initial condition x_0 and disturbances $d_1 \in \ell_\infty$, $d_2 \in \Omega_\infty$, $d_3 \in \ell_\infty(\delta_1)$.

Proof: We will denote $P(\varepsilon_a(\bar{x}))$ and $F(\varepsilon_a(\bar{x}))$ by P and F respectively to simplify notation. Define

$$\tilde{x} = x_b - \hat{x} = \begin{bmatrix} x_{1,n_1} - \hat{x}_1 \\ x_{2,n_2} - \hat{x}_2 \\ \vdots \\ x_{q,n_q} - \hat{x}_q \end{bmatrix}.$$

We have that

$$\tilde{x}^+ = \hat{A}\tilde{x} + \hat{B}\sigma(-\kappa\hat{B}'\hat{A}\tilde{x} - F\bar{x}) - \hat{B}\sigma(-F\bar{x}) + \hat{E}_2 d_2 + \hat{E}_3 d_3,$$

where

$$\hat{E}_j = \begin{bmatrix} E^j_{1,n_1} \\ E^j_{2,n_2} \\ \vdots \\ E^j_{q,n_q} \end{bmatrix}, \quad j = 2, 3. \tag{16}$$

Note that (A, B) is controllable implies that $(\hat{A}, \hat{B})$ is controllable. Moreover, $\hat{A}\hat{A}' = I$. The closed-loop system can be written in terms of $\tilde{x}, \bar{x}$ as

$$\begin{cases} \bar{x}^+ = A\bar{x} + B\sigma(-F\bar{x}) + \bar{E}_1 d_1 + \mathcal{I}\tilde{x} \\ \qquad \bar{B}\left[\sigma(-\kappa\hat{B}'\hat{A}\tilde{x} - F\bar{x}) - \sigma(-F\bar{x})\right] \\ \tilde{x}^+ = \hat{A}\tilde{x} + \hat{B}\sigma(-\kappa\hat{B}'\hat{A}\tilde{x} - F\bar{x}) - \hat{B}\sigma(-F\bar{x}) \\ \qquad +\hat{E}_2 d_2 + \hat{E}_3 d_3, \end{cases} \tag{17}$$

where $\bar{B}$ is B with B_{i,n_i} blocks set to zero and

$$\mathcal{I} = \begin{bmatrix} \mathcal{I}_1 \\ \mathcal{I}_2 \\ \vdots \\ \mathcal{I}_q \end{bmatrix}, \quad \mathcal{I}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \bar{I}_i \\ 0 \end{bmatrix}, \ \bar{I}_i = [0 \quad \cdots \quad \underset{\substack{\uparrow \\ i\text{th block}}}{I} \quad \cdots \quad 0].$$

We first prove global asymptotic stability without disturbances. Let $v = -F\bar{x}$. Our scheduling (15) guarantees that $\|v\| \leq \delta = \frac{1}{4}$ for any $\bar{x}$. Consider dynamics of $\tilde{x}$,

$$\tilde{x}^+ = \hat{A}\tilde{x} + \hat{B}\sigma(-\kappa\hat{B}'\hat{A}\tilde{x} + v) - \hat{B}\sigma(v).$$

Note that $8\kappa B'B \leq I$ implies that $8\kappa\hat{B}'\hat{B} \leq I$. Define a Lyapunov function as $V_1 = \tilde{x}^2$. Let $\chi = \kappa\hat{B}'\hat{A}\tilde{x}$. The increment of V_1 along the trajectories is given by

$$\begin{aligned} V_1^+ - V_1 &= [\sigma(-\chi + v) - \sigma(v)]'\hat{B}'\hat{B}[\sigma(-\chi + v) - \sigma(v)] \\ &\quad + \tfrac{2}{\kappa}\chi'[\sigma(-\chi + v) - \sigma(v)] \\ &\leq -\tfrac{1}{\kappa}\chi'\sigma(\chi) + \tfrac{1}{2\kappa}\|\sigma(\tfrac{1}{2}\chi)\|^2 \\ &\leq -\tfrac{1}{\kappa}\chi'\sigma(\chi) + \tfrac{1}{2\kappa}\|\sigma(\chi)\|^2 \\ &\leq -\tfrac{1}{2\kappa}\chi'\sigma(\chi) \\ &= -\tfrac{1}{2}\tilde{x}'\hat{A}'\hat{B}\sigma(\kappa\hat{B}'\hat{A}\tilde{x}) \end{aligned}$$

where we use that $8\kappa\hat{B}'\hat{B} \leq I$ and Lemma 2.

This clearly implies that $\hat{B}'\hat{A}\tilde{x}(k) \to 0$ as $k \to \infty$ and hence there exists a K_0 such that we have:

$$\|\kappa\hat{B}'\hat{A}\tilde{x}(k)\| \leq \tfrac{1}{2}$$

for $k \geq K_0$ and hence we obtain:

$$\tilde{x}^+ = (\hat{A} - \kappa \hat{B}\hat{B}'\hat{A})\tilde{x}$$

and therefore $\tilde{x}(k) \to 0$ as $k \to \infty$ because the matrix $\hat{A} - \kappa \hat{B}\hat{B}'\hat{A}$ is Schur stable.

For $k \geq K_0$, we have for $\bar{x}$ dynamics that

$$\bar{x}^+ = A\bar{x} + B\sigma(-F\bar{x}) + \bar{E}_1 d_1 + \bar{\mathcal{I}}\tilde{x}$$

where $\bar{\mathcal{I}} = \mathcal{I} - \kappa \bar{B}\hat{B}'\hat{A}$. Define $V_2 = \bar{x}' P \bar{x}$ and a set

$$\mathcal{K} = \left\{ \bar{x} \mid V_2(\bar{x}) \leq \alpha^2 \triangleq \frac{\delta^2}{b\,\mathrm{trace}(P^*)} \right\}$$

where P^* is the solution of (13) with $\varepsilon = 0.9$. It can be easily seen from (15) that for $\bar{x} \in \mathcal{K}$, $\varepsilon_a(\bar{x}) = 0.9$. Next, consider the increment of V_2 along the trajectory. There exists a β independent of d such that

$$\begin{aligned}
V_2^+ - V_2 &\leq -\varepsilon V_2 - 2\sigma(F\bar{x})' B' P \bar{\mathcal{I}}\tilde{x} + 2\bar{x}' A P \bar{\mathcal{I}}\tilde{x} \\
&\qquad + \tilde{x}'\bar{\mathcal{I}}' P \bar{\mathcal{I}}\tilde{x} + (\bar{x}^+)'[P^+ - P]\bar{x}^+ \\
&\leq -\varepsilon V_2 + 2\|A\|\sqrt{V_2}\|P^{1/2}\bar{\mathcal{I}}\tilde{x}\| + \beta\|P^{1/2}\bar{\mathcal{I}}\tilde{x}\| \\
&\qquad + \|P^{1/2}\bar{\mathcal{I}}\tilde{x}\|^2 + (\bar{x}^+)'[P^+ - P]\bar{x}^+
\end{aligned}$$

Note that $\mathcal{I}$, $\bar{B}$ and hence $\bar{\mathcal{I}}$ are in the form of (11). Lemma 1 shows that there exists an M such that

$$\|P^{1/2}\bar{\mathcal{I}}\tilde{x}\| = \sqrt{\tilde{x}'\bar{\mathcal{I}}' P \bar{\mathcal{I}}\tilde{x}} \leq \varepsilon\sqrt{M}\|\tilde{x}\|.$$

Moreover, for $\bar{x} \notin \mathcal{K}$, $V_2 \geq \alpha^2$. Hence we have for $\bar{x} \notin \mathcal{K}$,

$$\begin{aligned}
V_2^+ - V_2 &\leq -\varepsilon V_2 + 2\varepsilon\sqrt{MV_2}\|A\|\|\tilde{x}\| + \tfrac{\beta}{\alpha}\varepsilon\sqrt{MV_2}\|\tilde{x}\| \\
&\qquad + \varepsilon\tfrac{1}{\alpha}M\sqrt{V_2}\|\tilde{x}\|^2 + (\bar{x}^+)'[P^+ - P]\bar{x}^+ \\
&\leq -\varepsilon\sqrt{V_2}\Big[\sqrt{V_2} - 2\sqrt{M}\|A\|\|\tilde{x}\| - \tfrac{\beta}{\alpha}\sqrt{M}\|\tilde{x}\| \\
&\qquad - \tfrac{1}{\alpha}M\|\tilde{x}\|^2\Big] + (\bar{x}^+)'[P^+ - P]\bar{x}^+.
\end{aligned}$$

Since $\tilde{x} \to 0$, there exists a $K_1 > K_0$ such that for $k \geq K_1$,

$$\|\tilde{x}\| \leq \min\left\{1, \frac{\alpha^2}{4\sqrt{M}\|A\|\alpha + 2\beta\sqrt{M} + 2M}\right\}.$$

Therefore, for $k \geq K_1$ and $\bar{x} \notin \mathcal{K}$,

$$\begin{aligned}
\sqrt{V_2} - (2\sqrt{M}\|A\| - \tfrac{\beta}{\alpha}\sqrt{M})\|\tilde{x}\| - \tfrac{1}{\alpha}M\|\tilde{x}\|^2& \\
\geq \sqrt{V_2} - \tfrac{\alpha}{2} \geq \tfrac{\sqrt{V_2}}{2}&
\end{aligned}$$

and thus

$$V_2^+ - V_2 \leq -\tfrac{\varepsilon}{2}V_2 + (\bar{x}^+)'[P^+ - P]\bar{x}^+.$$

Since

$$V_2^+ - V_2 \quad \text{and} \quad (\bar{x}^+)'[P^+ - P]\bar{x}^+$$

cannot have the same sign (see [3]), we conclude that for $\bar{x} \notin \mathcal{K}$ and $k > K_1$,

$$V_2^+ - V_2 < 0.$$

This implies that $\bar{x}$ will enter $\mathcal{K}$ within finite time, say $K_2 > K_1$, For $k > K_2$ and $\bar{x} \in \mathcal{K}$, we have $\varepsilon = 0.9$ and $|\kappa\hat{B}'\hat{A}\tilde{x}| \leq \frac{1}{2}$. All saturations are inactive and the system becomes

$$\begin{cases} \bar{x}^+ = [A - BF^*]\,\bar{x} + \bar{\mathcal{I}}\tilde{x}, \\ \tilde{x}^+ = \left[\hat{A} - \kappa\hat{B}\hat{B}'\hat{A}\right]\tilde{x}, \end{cases}$$

where $F^* = (B'P^*B + I)^{-1}B'P^*A$ and P^* is the solution of (13) with $\varepsilon = 0.9$. It is clear that we shall also have $V_2^+ - V_2 < 0$ for $\bar{x} \in \mathcal{K}$. Therefore, $\bar{x}$ will remain in $\mathcal{K}$ for $k > K + K_0$. The global asymptotic stability follows from the properties that $A - BF^*$ and $\hat{A} - \kappa\hat{B}\hat{B}'\hat{A}$ are Schur stable with $8\kappa\hat{B}\hat{B}' \leq I$.

We proceed to show the boundedness of trajectories in presence of d_1, d_2 and d_3. We define

$$R = (\hat{A}')^k, \quad \text{and} \quad y = R\tilde{x}.$$

We note that since $\hat{A}'\hat{A} = I$, R defines a discrete-time rotation matrix. Moreover, we have that $R^+ = R\hat{A}'$. We obtain that

$$\begin{aligned}
y^+ = y + R^+\hat{B}\sigma(-\kappa\hat{B}'\hat{A}R'y + v) - R^+\hat{B}\sigma(v)& \\
+ R^+\hat{E}_2 d_2 + R^+\hat{E}_3 d_3&
\end{aligned}$$

with $y(0) = \tilde{x}_0$ where $v = -F\bar{x}$. Let $\bar{y}$ satisfy

$$\bar{y}^+ = \bar{y} + R^+\hat{E}_2 d_2, \quad \bar{y}(0) = \tilde{x}_0.$$

Since $d_2 \in \Omega_\infty$, we find that $\bar{y} \in \ell_\infty$. Define $\tilde{y} = y - \bar{y}$. Then

$$\begin{aligned}
y^+ = y + R^+\hat{B}\sigma(-\kappa\hat{B}'\hat{A}R'y - v) - R^+\hat{B}\sigma(v) + R^+\hat{E}_2 d_2& \\
+ \hat{E}_3 d_3, \quad y(0) = \tilde{x}(0),&
\end{aligned}$$

where $v = -F\bar{x}$. Let $\bar{y}$ satisfy

$$\bar{y}^+ = \bar{y} + R^+\hat{E}_2 d_2, \quad \bar{y}(0) = \tilde{x}_0.$$

Since $d_2 \in \Omega_\infty$, we find that $\bar{y} \in \ell_\infty$. Define $\tilde{y} = y - \bar{y}$. Then

$$\begin{aligned}
\tilde{y}^+ = \tilde{y} + R^+\hat{B}\sigma(-\kappa\hat{B}'\hat{A}R'\tilde{y} - \kappa\hat{B}'\hat{A}R'\bar{y} + v) - R^+\hat{B}\sigma(v)& \\
+ R^+\hat{E}_3 d_3, \quad \tilde{y}(0) = 0.&
\end{aligned}$$

Again define $z = R'\tilde{y}$. We get

$$z^+ = \hat{A}z + \hat{B}\sigma(-\kappa\hat{B}'\hat{A}z - u) - \hat{B}\sigma(v) + \hat{E}_3 d_3, \quad z(0) = 0,$$

where

$$u = \kappa\hat{B}'\hat{A}R'\bar{y} - v.$$

Consider an auxiliary system

$$w^+ = (\hat{A} + \hat{B}\hat{F})w + \hat{E}_3 d_3, \quad w(0) = 0,$$

where $\hat{F}$ is such that $\hat{A} + \hat{B}\hat{F}$ is Schur stable. Let δ be small enough such that $\|d_3\|_{\ell_\infty} \leq \delta$ implies that $\|\hat{F}w\|_{\ell_\infty} \leq 1/4$. Consider $\xi = z - w$. We get

$$\xi^+ = \hat{A}\xi + \hat{B}\sigma(-\kappa\hat{B}'\hat{A}\xi - \hat{u}) - \hat{B}\sigma(v) - \hat{B}\hat{F}w$$

where $\hat{u} = \kappa\hat{B}'\hat{A}w + \kappa\hat{B}'\hat{A}R'\bar{y} - v$. Since $\hat{u} \in \ell_\infty$ and $\|v\| + \|\hat{F}w\| \leq 1/4 + 1/4 = 1/2$, it follows from Lemma 2 in [6] that $\xi \in \ell_\infty$ for sufficiently small δ. This implies that $\tilde{x} \in \ell_\infty$.

Consider the dynamics of $\bar{x}$

$$\bar{x}^+ = A\bar{x} + B\sigma(-F\bar{x}) + \bar{B}\zeta + \bar{E}_1 d_1 + \mathcal{I}\tilde{x},$$

where $\zeta = \sigma(-\kappa \hat{B}'\hat{A}\tilde{x} - F\bar{x}) - \sigma(F\bar{x})$. Because $\sigma(\cdot)$ is globally Lipschitz with Lipschitz constant 1, $\|\zeta\| \leq \|\kappa \hat{B}'\hat{A}\tilde{x}\|$. Hence $\zeta \in \ell_\infty$. There exists a β such that

$$\begin{aligned}
&V_2^+ - V_2 \\
&\leq -\varepsilon V_2 - 2\sigma(F\bar{x})'B'P\mathcal{I}\tilde{x} - 2\sigma(F\bar{x})'B'P\bar{E}_1 d_1 \\
&\qquad - 2\sigma(F\bar{x})'BP\bar{B}\zeta + 2\bar{x}'A'P\bar{B}\zeta \\
&\qquad + 2\bar{x}'A'P\bar{E}_1 d_1 + 2\bar{x}'A'P\mathcal{I}\tilde{x} \\
&\qquad + (d_1'\bar{E}_1' + \tilde{x}'\mathcal{I}' + \zeta'\bar{B}')P(\bar{E}_1 d_1 + \mathcal{I}\tilde{x} + \bar{B}\zeta) \\
&\qquad + (\bar{x}^+)'[P^+ - P]\bar{x}^+ \\
&\leq -\varepsilon V_2 + \Big(\|P^{1/2}\mathcal{I}\tilde{x}\| + \|P^{1/2}\bar{E}_1 d_1\| + \|P^{1/2}\bar{B}\zeta\|\Big)^2 \\
&\qquad + (2\|A\|\sqrt{V_2} + \beta)\Big(\|P^{1/2}\mathcal{I}\tilde{x}\| + \|P^{1/2}\bar{E}_1 d_1\| \\
&\qquad + \|P^{1/2}\bar{B}\zeta\|\Big) + (\bar{x}^+)'[P^+ - P]\bar{x}^+
\end{aligned}$$

We have already shown that according to Lemma 1 there exist M, M_1 and M_2 such that

$$\|P^{1/2}\mathcal{I}\tilde{x}\| \leq \varepsilon\sqrt{M}\|\tilde{x}\|, \quad \|P^{1/2}\bar{E}_1 d_1\| \leq \varepsilon\sqrt{M_1}\|d_1\|, \\ \|P^{1/2}\bar{B}\zeta\| \leq \varepsilon\sqrt{M_2}\|\zeta\|$$

Define set $\mathcal{V}$ as $\mathcal{V} = \{\bar{x} \mid V_2(\bar{x}) \leq c\}$ where c is such that $V_2 \geq c$ implies that

$$\begin{aligned}
&\tfrac{1}{2}V_2 \geq (2\|A\|\sqrt{V_2} + \beta)(\sqrt{M}\|\tilde{x}\|_\infty + \sqrt{M_1}\|d_1\|_\infty \\
&+ \sqrt{M_2}\|\zeta\|_\infty) + 2M\|\tilde{x}\|_\infty^2 + 2M_1\|d_1\|_\infty^2 + 2M_2\|\zeta\|_\infty^2
\end{aligned}$$

Therefore, for $\bar{x} \notin \mathcal{V}$, we have

$$V_2^+ - V_2 \leq -\tfrac{1}{2}\varepsilon V_2 + (\bar{x}^+)'[P^+ - P]\bar{x}^+.$$

Since $V_2^+ - V_2$ and $(\bar{x}^+)'[P^+ - P]\bar{x}^+$ can not have the same sign, we find that

$$V_2^+ - V_2 < 0, \quad \bar{x} \notin \mathcal{V}. \tag{18}$$

On the other hand, for $\bar{x} \in \mathcal{V}$, suppose $V_2^+ - V_2 \geq 0$. We first have that $(\bar{x}^+)'[P^+ - P]\bar{x}^+ \leq 0$. Then

$$\begin{aligned}
&V_2^+ - V_2 \leq \Delta := 2M\|\tilde{x}\|_\infty^2 + 2M_1\|d_1\|_\infty^2 + 2M_2\|\zeta\|_\infty^2 \\
&+ (2\|A\|\sqrt{c} + \beta)(\sqrt{M}\|\tilde{x}\|_\infty + \sqrt{M_1}\|d_1\|_\infty + \sqrt{M_2}\|\zeta\|_\infty)
\end{aligned}$$

Hence the maximum increment of V_2 inside $\mathcal{V}$ is Δ. In view of this, (18) and definition of $\mathcal{V}$, we conclude that $V_2 \leq \max\{c + \Delta, V_2(0)\}$, which, by the property (5) of scheduling (15), implies that $\bar{x} \in \ell_\infty$ and hence $x \in \ell_\infty$. ∎

Appendix

We have used the following inequalities in the paper:

Lemma 2: For two vectors $s, t \in \mathbb{R}^m$, the following statements hold:

1) $|s'[\sigma(s + t) - \sigma(s)]| \leq 2\sqrt{m}\|t\|$;
2) if $\|t\| \leq \frac{1}{2}$, then $2s'[\sigma(t) - \sigma(t - s)] \geq s'\sigma(s)$;
3) $\|s - \sigma(s)\| \leq s'\sigma(s)$;
4) For $\|t\| \leq 1$, $\|\sigma(s + t) - \sigma(t)\| \leq 2\|\sigma(\frac{1}{2}s)\|$.

Proof: (1)-(3) have been proved in [4]. Consider (4). Let s_i and t_i denote each element of s and t.

Case 1: $u_i + v_i \geq 1$, we have that

$$|\sigma(s_i + t_i) - \sigma(t_i)| = 1 - t_i \leq 2.$$

Also

$$|\sigma(s_i + t_i) - \sigma(t_i)| = 1 - t_i \leq |u_i|.$$

Hence

$$|\sigma(s_i + t_i) - \sigma(t_i)| \leq 2|\sigma(\tfrac{1}{2}s_i)|.$$

Case 2: $|s_i + t_i| < 1$, this implies that $|s_i| \leq 2$,

$$|\sigma(s_i + t_i) - \sigma(t_i)| = |s_i| = 2|\sigma(\tfrac{1}{2}s_i)|.$$

Case 3: $u_i + v_i \leq -1$

$$|\sigma(s_i + t_i) - \sigma(t_i)| = |-1 - t_i| = 1 + t_i \leq |s_i|.$$

Also

$$|\sigma(s_i + t_i) - \sigma(t_i)| = |-1 - t_i| = 1 + t_i \leq 2.$$

Hence

$$|\sigma(s_i + t_i) - \sigma(t_i)| \leq 2|\sigma(\tfrac{1}{2}s_i)|.$$

∎

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