Consensus in the network with uniform constant communication delay

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Abstract

This paper studies consensus among identical agents that are at most critically unstable and coupled through networks with uniform constant communication delay. An upper bound for delay tolerance is obtained which explicitly depends on agent dynamics and network topology. The dependence on network topology disappears in the case of undirected networks. For any delay satisfying this upper bound, a controller design methodology without exact knowledge of the network topology is proposed so that multi-agent consensus in a set of unknown networks can be achieved. Moreover, when the network topology is known, a larger delay tolerance is possible via a topology-dependent consensus controller.

1 Introduction

The consensus problem in a network has received substantial attention in recent years, partly due to the wide applications in areas such as sensor networks and autonomous vehicle control. A relatively complete coverage of earlier work can be found in the survey paper of Olfati-Saber et al. (2007), the recent books by Wu (2007); Ren & Y.C. Cao (2011) and references therein.

Consensus in a network with time delay has been extensively studied in the literature. Most results consider the agent model as described by single-integrator dynamics (Bliman & Ferrari-Trecate, 2008; Tian & Liu, 2008; Olfati-Saber & Murray, 2004), or double-integrator dynamics (Tian & Liu, 2009; Lin et al., 2007; Yu et al., 2010). Specifically, it is shown by Olfati-Saber & Murray (2004) that a network of single-integrator agents subject to uniform constant communication delay can achieve consensus with a particular linear local control protocol if and only if the delay is bounded by a maximum that is inversely proportional to the largest eigenvalue of the graph Laplacian associated with the network. This result was later on generalized in Bliman & Ferrari-Trecate (2008) to non-uniform constant or time-varying delays. Sufficient conditions for consensus among agents with first order dynamics were also obtained in Tian & Liu (2008). The results in Olfati-Saber & Murray (2004) were extended in Lin et al. (2007); Yu et al. (2010) to double integrator dynamics. An upper bound on the maximum network delay tolerance for second-order consensus of multi-agent systems with any given linear control protocol was obtained.

In this paper, we study the multi-agent consensus problem with uniform constant communication delay. The agents are assumed to be multi-input and multi-output and at most critically unstable, i.e. each agent has all its eigenvalues in the closed left half plane. In other words, we allow the agents to have eigenvalues on the imaginary axis. The contribution of this paper with respect to aforementioned literature is twofold: first, we find a sufficient condition on the tolerable communication delay for agents with high-order dynamics, which has an explicit dependence on the agent dynamics and network topology. For undirected networks, this upper bound can be independent of network topology provided that the network is connected. Moreover, in a special case where the agents only have zero eigenvalues, such as single- and double-integrator dynamics, arbitrarily large but bounded delay can be tolerated. Another layer of contribu-

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tion is that for delays satisfying the proposed upper bound, we present a controller design methodology without precise knowledge of network topology so that the multi-agent consensus in a set of unknown networks can be achieved. When the network topology is accurately known, the controller design can be modified to be topology-dependent so that a larger delay tolerance is attainable.

Münz et al. (2010, 2012) have recently presented interesting results on robust consensus of linear multi-agent systems (MAS) subject to heterogeneous feedback delays. These works are more general and realistic in respect of diverse delays. However, this paper extends Münz et al. (2010, 2012) in the following ways. First of all, Münz et al. (2010, 2012) study the consensus problem in undirected networks. With the design proposed in this paper, we are able to achieve consensus in a set of directed networks. Secondly, Münz et al. (2010, 2012) only consider single-input and single-output agents whose eigenvalues are in the open left half plane, except for those at the origin. We consider multi-input and multi-output agents that have eigenvalues in the closed left half plane. In other words, eigenvalues on the imaginary axis are also permitted.

The rest of the paper is organized as follows: notations and some preliminary results are presented in the remainder of Section 1. System and network configuration and consensus problem formulations are given in Section 2. The consensus problem with full-state coupling is solved in Section 3. The corresponding problem with partial-state coupling is dealt with in Section 4. In Section 5, we discuss the special case of neutrally stable systems. Some technical lemmas are appended at the end of this paper.

The following notations will be used in this paper. For a vector d, we denote a diagonal matrix by D=diag$\{d\}$ whose diagonal is specified by d. For column vectors $x_1,\dots,x_n$, the stacking column vector of $x_1,\dots,x_n$ is denoted by $[x_1;\dots;x_n]$.

A weighted graph G defined by a pair $(\mathcal{N},\mathcal{E})$ contains a *directed spanning tree* if there is a node $r\in\mathcal{N}$ such that a directed path exists between r and any other node. For a weighted graph $G(\mathcal{N},\mathcal{A})$ with $\mathcal{A}=\{a_{ij}\}_{N\times N}$, a matrix $L=\{\ell_{ij}\}_{N\times N}$ with

$$\ell_{ij}=\begin{cases}\sum_{j=1}^{N}a_{ij}, & i=j\\ -a_{ij}, & i\neq j,\end{cases}$$

is called the *Laplacian matrix* associated with graph G. In the case where G has non-negative weights, L has all its eigenvalues in the closed right half plane and at least one eigenvalue at zero associated with right eigenvector $\mathbf{1}$ (see Godsil & Royle, 2001). If G has a directed spanning tree, L has a simple eigenvalue at zero and all the other eigenvalues have strictly positive real parts (see e.g. Ren & Beard, 2005).

2 Problem formulation

Consider a network of N identical agents

$$\begin{cases}\dot{x}^i(t)=Ax^i(t)+Bu^i(t), \quad i=1,\dots,N,\\ z^i(t)=\sum_{j=1}^{N}\ell_{ij}x^j(t-\tau).\end{cases}\tag{1}$$

where $x^i\in\mathbb{R}^n$, $u^i\in\mathbb{R}^m$ and $z^i\in\mathbb{R}^n$, τ is an unknown constant satisfying $\tau\in[0,\bar{\tau}]$. The coefficients ℓ_{ij} are such that $\ell_{ij}\leq 0$ for $i\neq j$ and $\ell_{ii}=-\sum_{j\neq i}^{N}\ell_{ij}$. In (1), each agent collects a delayed measurement z^i of the state of neighboring agents through the network, which we refer to as *full-state coupling*.

It is also common that z^i may consist of the outputs of neighboring agents instead of the complete states which can be formulated as follows:

$$\begin{cases}\dot{x}^i(t)=Ax^i(t)+Bu^i(t),\\ y^i(t)=Cx^i(t), \qquad i=1,\dots,N,\\ z^i(t)=\sum_{j=1}^{N}\ell_{ij}y^j(t-\tau),\end{cases}\tag{2}$$

where $x^i\in\mathbb{R}^n$, $u^i\in\mathbb{R}^m$ and $y^i,z^i\in\mathbb{R}^p$. We refer to the agents in this case as having *partial-state coupling*.

The matrix $L=\{\ell_{ij}\}\in\mathbb{R}^{N\times N}$ defines a *communication topology* that can be captured by a weighted graph $G=(\mathcal{N},\mathcal{E})$ where $(j,i)\in\mathcal{E}\Leftarrow\ell_{ij}<0$. The graph G is, in general, directed. However, in a special case where L is symmetric, G is undirected. This L is the Laplacian matrix associated with G.

Assumption 1 *The following assumptions are made throughout the paper:*

(1) The agents are at most critically unstable, that is, A has all its eigenvalues in the closed left half plane;
(2) (A,B) is stabilizable and (A,C) is detectable;
(3) The communication topology described by the graph G contains a directed spanning tree.

Under item 3 of Assumption 1, L has one simple eigenvalue at zero and the others lie in the open right half plane. Let $\lambda_1,\cdots,\lambda_N$ denote the eigenvalues of L and assume $\lambda_1=0$. We have that $\mathrm{Re}(\lambda_i)>0$, or equivalently $\arg(\lambda_i)\in(-\frac{\pi}{2},\frac{\pi}{2})$, for $i=2,...,N$.

It should be noted that in practice, perfect information of the communication topology is usually not available for controller design and that only some rough characterization of the network can be obtained. Using the non-zero eigenvalues of L as a "measure" for the graph, we can introduce the following definition to characterize a set of unknown communication topologies.

Definition 1 *For any $\gamma \geq \beta \geq 0$ and $\frac{\pi}{2} > \varphi \geq 0$, $\mathcal{G}_{\beta,\gamma,\varphi}$ is the set of graphs satisfying Assumption 1 and whose associated Laplacian satisfies*

$$|\lambda_i| \in (\beta, \gamma) \text{ and } \arg \lambda_i \in [-\varphi, \varphi]$$

for $i = 2, \ldots, N$.

Remark 1 *In the literature, only the real part of the Laplacian eigenvalues is typically of concern when studying synchronization. The smallest non-zero real part is the algebraic connectivity of the graph, which plays an important role in most work on network synchronization (e.g Li et al., 2010). The largest magnitude depends on the topology of the graph and the numerical edge weights, and plays a role in some synchronization problems (see, for instance, Olfati-Saber & Murray, 2004; Seo et al., 2009a; Mesbahi & Egerstedt, 2010). For undirected graphs, all the eigenvalues of the Laplacian are real. In this case, φ in Definition 1 can be taken as zero, and β and γ become traditional bounds on the non-zero real parts. For directed graphs, however, the argument of the Laplacian eigenvalues can vary in the range $(-\pi/2, \pi/2)$. An interesting insight offered by this paper is that the magnitude of this argument plays a crucial role in the presence of communication delay. Consequently, Definition 1 delineates sets of graphs based both on the magnitude and argument of the Laplacian eigenvalues.*

Definition 2 *The agents in the network achieve consensus if*

$$\lim_{t \to \infty} (x^i(t) - x^j(t)) = 0, \quad \forall i, j \in \{1, \ldots, N\}.$$

Two consensus problems for agents with full-state coupling (1) and partial-state coupling (2) can be formulated for this set of networks respectively as follows.

Problem 1 *Consider a network of agents* (1) *with full state coupling. The consensus problem, given a set of possible communication topologies $\mathcal{G}_{\beta,\gamma,\varphi}$ and a delay upper bound $\bar{\tau}$, is to design linear static controllers $u^i = Fz^i$ for $i = 1, \ldots, N$ such that the agents* (1) *with $u^i = Fz^i$ achieve consensus with any communication topology belonging to $\mathcal{G}_{\beta,\gamma,\varphi}$ and for $\tau \leq \bar{\tau}$.*

Problem 2 *Consider a network of agents* (2) *with partial state coupling. The consensus problem with a set of possible communication topologies $\mathcal{G}_{\beta,\gamma,\varphi}$ and a delay upper bound $\bar{\tau}$ is to design linear dynamic control protocols of the form:*

$$\begin{cases} \dot{\chi}^i = A_c \chi^i + B_c z^i \\ u^i = C_c \chi^i, \end{cases} \tag{3}$$

for $i = 1, \ldots, N$ such that the agents (2) *with controller* (3) *achieve consensus with any communication topology belonging to $\mathcal{G}_{\beta,\gamma,\varphi}$ and for $\tau \leq \bar{\tau}$.*

3 Consensus with full-state coupling

In this section, we consider agents with full-state coupling as given in (1) and solve Problem 1.

For a given set of networks $\mathcal{G}_{\beta,\gamma,\varphi}$, we design a decentralized local consensus controller for any network in $\mathcal{G}_{\beta,\gamma,\varphi}$ as follows:

$$u^i = -\alpha B' P_\varepsilon z^i. \tag{4}$$

Here P_ε is the positive definite solution of the Algebraic Riccati Equation (ARE)

$$A' P_\varepsilon + P_\varepsilon A - P_\varepsilon B B' P_\varepsilon + \varepsilon I = 0. \tag{5}$$

and ε, as well as α, are design parameters which will be chosen according to β, γ and φ so that the multi-agent consensus can be achieved with any communication topology belonging to $\mathcal{G}_{\beta,\gamma,\varphi}$. Let

$$\omega_{\max} = \begin{cases} 0, & A \text{ is Hurwitz.} \\ \max\{\omega \in \mathbb{R} \mid \det(j\omega I - A) = 0\}, & \text{otherwise.} \end{cases}$$

Theorem 1 *For a given set $\mathcal{G}_{\beta,\gamma,\varphi}$ with $\beta > 0$ and $\bar{\tau} > 0$, consider the agents* (1) *and any coupling network belonging to the set $\mathcal{G}_{\beta,\gamma,\varphi}$. In that case Problem 1 is solvable if,*

$$\bar{\tau} \omega_{\max} < \tfrac{\pi}{2} - \varphi. \tag{6}$$

Moreover, it can be solved by the consensus controller (4) *if* (6) *holds. Specifically, for given $\mathcal{G}_{\beta,\gamma,\varphi}$ and given $\bar{\tau}$ satisfying* (6), *there exist $\alpha > 0$ and $\varepsilon^* > 0$ such that for this α and any $\varepsilon \in (0, \varepsilon^*]$, the agents* (1) *with controller* (4) *achieve consensus for any communication topologies in $\mathcal{G}_{\beta,\gamma,\varphi}$ and $\tau \in [0, \bar{\tau}]$.*

Remark 2 *The philosophy underlying the proof can be briefed as follows: First, the consensus problem can be converted to a robust stabilization problem with both input delay and another layer of input uncertainty introduced by the unknown complex eigenvalues of the Laplacian matrix. Then we show that the low-gain feedback can accommodate these uncertainties if* (6) *is satisfied, by providing an infinite gain margin and a phase margin that can be made arbitrarily close to $\pi/2$. In fact, we can see in condition* (6) *that $\bar{\tau}\omega_{\max}$ characterize the delay effect and φ represents the topology uncertainty. Together they should not exceed $\pi/2$.*

Proof: The proof proceeds in two steps. Step 1: it follows from Lemma A.1 in the Appendix that Theorem 1 holds if for any $\gamma \geq \beta > 0$, $\bar{\tau} > 0$ and φ satisfying (6), there exist $\alpha > 0$ and ε^* such that for $\varepsilon \in (0, \varepsilon^*]$, the system

$$\dot{x} = Ax - \lambda \alpha e^{j\psi} B B' P_\varepsilon x(t - \tau). \tag{7}$$

is asymptotically stable for all $\tau \in [0, \bar{\tau}]$, $\lambda \in (\beta, \gamma)$ and $\psi \in [-\varphi, \varphi]$.

Since $\bar{\tau}$ and φ satisfy condition (6), this implies $\bar{\tau}\omega_{\max}+\varphi < \frac{\pi}{2}$. Choose α such that

$$\alpha > \frac{1}{\beta\cos(\varphi+\omega_{\max}\bar{\tau})}. \tag{8}$$

Let this α be fixed. (8) implies that $\alpha\lambda\cos(\varphi) > 1$ and hence, by Lemma C.1, $A-\alpha\lambda e^{j\psi}BB'P_\varepsilon$ is Hurwitz stable for $\psi\in[-\varphi,\varphi]$. Then it follows from Lemma B.1 that system (7) is asymptotically stable if

$$\det\Big[j\omega I - A + \alpha\lambda e^{j(\psi-\omega\tau)}BB'P_\varepsilon\Big] \neq 0, \tag{9}$$

for $\omega\in\mathbb{R}$, $\tau\in[0,\bar{\tau}]$ and $\psi\in[-\varphi,\varphi]$.

Step 2. We need to prove (9). We note that given (8), there exists a $\delta > 0$ such that

$$\alpha > \frac{1}{\beta\cos(\varphi+\omega\bar{\tau})}, \quad \forall|\omega| < \omega_{\max}+\delta. \tag{10}$$

Next we will split the proof of (9) in two cases where $|\omega| < \omega_{\max}+\delta$ and $|\omega| \geq \omega_{\max}+\delta$ respectively.

If $|\omega|\geq\omega_{\max}+\delta$, we have $\det(j\omega I - A)\neq 0$, which yields $\underline{\sigma}(j\omega I - A) > 0$. Hence, there exists $\mu > 0$ such that

$$\underline{\sigma}(j\omega I - A) > \mu, \quad \forall\omega, \text{ s.t. } |\omega|\geq\omega_{\max}+\delta.$$

To see this, note that for ω satisfying $|\omega| > \bar{\omega} := \max\{\|A\|+1, \omega_{\max}+\delta\}$, we have $\underline{\sigma}(j\omega I - A) > |\omega| - \|A\| > 1$. But for ω with $|\omega|\in[\omega_{\max}+\delta,\bar{\omega}]$, there exists $\mu\in(0,1]$ such that $\underline{\sigma}(j\omega I - A)\geq\mu$, which is due to the fact that $\underline{\sigma}(j\omega I - A)$ depends continuously on ω.

Given α and $\lambda\in(\beta,\gamma)$, there exists $\varepsilon^* > 0$ such that $\|\lambda\alpha BB'P_\varepsilon\|\leq\mu/2$ for $\varepsilon<\varepsilon^*$. Then

$$\underline{\sigma}(j\omega I - A - \alpha\lambda e^{j(\psi-\omega\tau)}BB'P_\varepsilon)\geq\mu-\mu/2\geq\mu/2.$$

Therefore, condition (9) holds for $|\omega|\geq\omega_{\max}+\delta$.

It remains to verify (9) with $|\omega|<\omega_{\max}+\delta$. By the definition of δ, we find that

$$\alpha\lambda\cos(\psi-\omega\tau) > \alpha\beta\cos(\varphi+|\omega|\bar{\tau}) > 1,$$

and hence by Lemma C.1, $A-\alpha\lambda e^{j(\psi-\omega\tau)}BB'P_\varepsilon$ is Hurwitz stable, for $\omega\in(-\omega_{\max}-\delta,\omega_{\max}+\delta)$, $\lambda\in(\beta,\gamma)$, $\psi\in[-\varphi,\varphi]$ and $\tau\in[0,\bar{\tau}]$ (See Fig. 1). Therefore, (9) also holds with $|\omega|<\omega_{\max}+\delta$. ∎

Remark 3 *Some comments on implementation of the consensus controller* (4) *are worthwhile. Four parameters are chosen sequentially in the consensus design and analysis, namely α, δ, μ and ε. First, we select the scaling parameter α in* (8) *using the given data β, φ and $\omega_{\max}$. Then, δ is chosen based on network data and the choice of α and such a δ will yield corresponding value of μ. Eventually, ε is determined by μ and γ.*

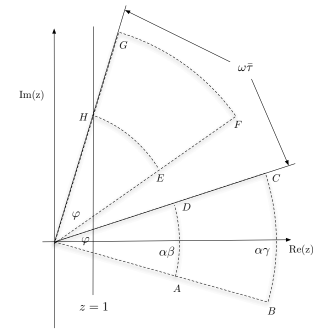

Fig. 1. Note that given the condition in Theorem 1, $\alpha\lambda e^{j\psi}$ is originally located in $ABCD$ and $\alpha\lambda e^{j(\psi-\omega\bar{\tau})}$ will be located in $EFGH$. It is easy to verify that if (10) holds, $EFGH$ will not cross the vertical line $z=1$ for $|\omega|<\omega_{\max}+\delta$. Therefore, $\mathrm{Re}(\alpha\lambda e^{j\psi-\omega\bar{\tau}}) > 1$.

Remark 4 *The consensus controller design depends only on the agent model and parameters $\bar{\tau},\beta,\gamma$ and φ and is independent of specific network topology provided that the network satisfies Assumption 1.*

In the special case where $\omega_{\max}=0$, i.e. the eigenvalues of A are either zero or in the open left half plane, then arbitrarily bounded communication delay can be tolerated as formulated in the following corollary:

Corollary 1 *For a given set $\mathcal{G}_{\beta,\gamma,\varphi}$ with $\beta>0$ and $\bar{\tau}>0$, consider the agents* (1) *and any communication topology belonging to the set $\mathcal{G}_{\beta,\gamma,\varphi}$. Suppose the eigenvalues of A are either zero or in the open left half plane. In that case, Problem 1 is always solvable via the consensus controller* (4). *Specifically, for given $\mathcal{G}_{\beta,\gamma,\varphi}$ and $\bar{\tau}>0$, there exist α and ε^* such that for any $\varepsilon\in(0,\varepsilon^*]$, the agents* (1) *with controller* (4) *achieve consensus for any communication topologies in $\mathcal{G}_{\beta,\gamma,\varphi}$ and $\tau\in[0,\bar{\tau}]$.*

Remark 5 *In previous studies of network consensus problem, agents are normally assumed to have single- or double-integrator type dynamics. Based on Corollary 1, we find that the delay tolerance in such cases is independent of network topology and can be made arbitrarily large. This result in no way contradicts that in Olfati-Saber & Murray (2004); Lin et al. (2007); Yu et al. (2010) since the goal here is to find the maximal achievable delay tolerance by controller design whereas the mentioned paper present conditions such that the delay does not affect consensus for a certain given controller.*

4 Consensus with partial-state coupling

Next, we consider the case of partial-state coupling and design a controller of the form (3) which solves Problem 2.

For $\varepsilon > 0$, let P_ε be the positive definite solution of the ARE (5). A dynamic low-gain consensus controller, which is a special form of (3), can be constructed as

$$\begin{cases} \dot{\chi}^i = (A + KC)\chi^i - Kz^i \\ u^i = -\alpha B' P_\varepsilon \chi^i, \end{cases} \tag{11}$$

where K is such that $A + KC$ is Hurwitz stable. α and ε are design parameters to be chosen later. We shall prove that this consensus controller solves Problem 2.

Theorem 2 *For a given set $\mathcal{G}_{\beta,\gamma,\varphi}$ with $\beta > 0$ and $\bar{\tau} > 0$, consider the agents* (2) *with any communication topology belonging to $\mathcal{G}_{\beta,\gamma,\varphi}$. In that case, Problem 2 is solvable if,*

$$\bar{\tau}\omega_{\max} < \tfrac{\pi}{2} - \varphi. \tag{12}$$

Moreover, it can be solved by the consensus controller (11) *if* (12) *holds. Specifically, for given β and γ and given φ and $\bar{\tau}$ satisfying* (12), *there exist $\alpha > 0$ and ε^* such that for any $\varepsilon \in (0, \varepsilon^*]$, the agents* (2) *with controller* (11) *achieve consensus for any communication topology in $\mathcal{G}_{\beta,\gamma,\varphi}$ and $\tau \in [0, \bar{\tau}]$.*

Proof: It follows from Lemma A.2 in the Appendix that Theorem 2 holds if there exist $\alpha > 0$ and $\varepsilon^* > 0$ such that for $\varepsilon \in (0, \varepsilon^*]$, the system

$$\begin{cases} \dot{x}(t) = Ax(t) - \alpha\lambda e^{j\psi} BB' P_\varepsilon \chi(t - \tau) \\ \dot{\chi}(t) = (A + KC)\chi(t) - KCx(t) \end{cases} \tag{13}$$

is asymptotically stable for any $\lambda \in (\beta, \gamma)$, $\psi \in [-\varphi, \varphi]$ and $\tau \in [0, \bar{\tau}]$.

Define

$$\bar{\mathcal{A}} = \begin{bmatrix} A & 0 \\ -KC & A + KC \end{bmatrix}, \quad \bar{\mathcal{B}} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \bar{\mathcal{F}}_\varepsilon = \begin{bmatrix} 0 & -B' P_\varepsilon \end{bmatrix}.$$

First of all, for given β, γ and $\psi \in (-\varphi, \varphi)$, there exists α such that

$$\alpha > \frac{1}{\beta \cos(\varphi + \omega_{\max}\bar{\tau})}. \tag{14}$$

Let this α be fixed. By Lemma C.2 in the Appendix, there exists ε_1 such that for $\varepsilon \in (0, \varepsilon_1]$, $\bar{\mathcal{A}} + \alpha\lambda e^{j\psi} \bar{\mathcal{B}}\bar{\mathcal{F}}_\varepsilon$ is Hurwitz stable for $\lambda \in (\beta, \gamma)$ and $\psi \in (-\varphi, \varphi)$. It follows from Lemma B.1 that (13) is asymptotically stable if

$$\det\left[j\omega I - \bar{\mathcal{A}} - \alpha\lambda e^{j(\psi - \omega\tau)} \bar{\mathcal{B}}\bar{\mathcal{F}}_\varepsilon\right] \neq 0, \ \forall \omega \in \mathbb{R}, \\ \forall \lambda \in (\beta, \gamma), \ \forall \psi \in (-\varphi, \varphi), \ \forall \tau \in [0, \bar{\tau}]. \tag{15}$$

Given (14), there exists $\delta > 0$ such that

$$\lambda\alpha \cos(\varphi + \omega\bar{\tau}) > 1, \quad \forall |\omega| < \omega_{\max} + \delta. \tag{16}$$

We can show, as in the proof of Theorem 1, that there exists $\varepsilon_2 \leq \varepsilon_1$ such that for $\varepsilon \in (0, \varepsilon_2]$, condition (15) holds for $|\omega| \geq \omega_{\max} + \delta$.

For $|\omega| < \omega_{\max} + \delta$, it follows from (16) and Lemma C.2 that $\bar{\mathcal{A}} + \alpha\lambda e^{j(\psi - \omega\tau)} \bar{\mathcal{B}}\bar{\mathcal{F}}_\varepsilon$ is Hurwitz stable. Therefore, condition (15) also holds with $|\omega| < \omega_{\max} + \delta$. ∎

Remark 6 *The low-gain compensator* (11) *is constructed based on the agent model and the network characteristics β, γ and φ. The four parameters α, δ, μ and ε used in the design of controller* (11) *are chosen with the same order and relation as in the proof of Theorem 1.*

Corollary 2 *For a given set $\mathcal{G}_{\beta,\gamma,\varphi}$ with $\beta > 0$ and $\bar{\tau} > 0$, consider the agents* (2) *with any communication topology belonging to $\mathcal{G}_{\beta,\gamma,\pi}$. Suppose the eigenvalues of A are either zero or in the open left half plane. In that case, Problem 2 is solvable by the consensus controller* (11). *Specifically, for given β, γ, φ and $\bar{\tau} > 0$, there exist $\alpha > 0$ and $\varepsilon^* > 0$ such that for any $\varepsilon \in (0, \varepsilon^*]$, the agents* (2) *with controller* (11) *achieve consensus for any communication topology in $\mathcal{G}_{\beta,\gamma,\varphi}$ and $\tau \in [0, \bar{\tau}]$.*

5 Special case: Neutrally stable agents

We observe that the consensus controller design in Theorem 1 and Theorem 2 for general critically unstable agents depends on β which is related to the algebraic connectivity of the graph. We next consider a special case where the agent dynamics are neutrally stable, that is, the eigenvalues of A on the imaginary axis, if any, are semi-simple. Without loss of generality, we assume that $A' + A \leq 0$ which can be obtained after a suitable basis transformation. In this case, we shall show that the consensus controller design no longer requires the knowledge of β and hence allows us to deal with a larger set of unknown communication topologies that can be denoted as $\mathcal{G}_{0,\gamma,\varphi}$.

Consider the agents (1). Assume $A' + A \leq 0$. A local consensus controller can be constructed as

$$u^i = \varepsilon B' z^i. \tag{17}$$

We have the following theorem:

Theorem 3 *For a given set $\mathcal{G}_{0,\gamma,\varphi}$ and $\bar{\tau} > 0$, consider the agents* (1) *and any communication topology belonging to the set $\mathcal{G}_{0,\gamma,\varphi}$. Suppose $A' + A \leq 0$. In that case, Problem 1 is solvable if,*

$$\bar{\tau}\omega_{\max} < \tfrac{\pi}{2} - \varphi, \tag{18}$$

Moreover, it can be solved by the consensus controller (17) *if* (18) *holds. Specifically, for given γ and given φ and $\bar{\tau}$*

satisfying (18)*, there exists an* ε^* *such that for any* $\varepsilon \in (0, \varepsilon^*]$*, the agents* (1) *with controller* (17) *achieve consensus for any communication topology in* $\mathcal{G}_{0,\gamma,\varphi}$ *and* $\tau \in [0, \bar{\tau}]$.

Proof: It follows from Lemma A.1 that Theorem 3 holds if the system

$$\dot{x} = Ax - \lambda\varepsilon e^{j\psi} BB' x(t-\tau) \tag{19}$$

is asymptotically stable for $\lambda \in (0, \gamma)$, $\psi \in [-\varphi, \varphi]$ and $\tau \in [0, \bar{\tau}]$, which, by Lemma B.1, is true if and only if

$$\det\left[j\omega I - A + \varepsilon\lambda e^{j\psi - j\omega\tau} BB'\right] \neq 0, \quad \forall \omega \in \mathbb{R},\ \lambda \in (0,\gamma),\ \psi \in [-\varphi,\varphi],\ \tau \in [0,\bar{\tau}]. \tag{20}$$

There exists $\delta > 0$ such that $\omega\bar{\tau} + \varphi < \frac{\pi}{2}$, $\forall \omega$ s.t. $|\omega| < \omega_{\max} + \delta$.

For given $\lambda \in (0, \gamma)$, we can show with a similar argument as in the proof of Theorem 1 that there exists a $\mu > 0$ and a ε_1 such that for $\varepsilon \in (0, \varepsilon_1]$ and $\lambda \in (0, \gamma)$

$$\underline{\sigma}(j\omega I - A + \varepsilon\lambda e^{j\psi - j\omega\tau} BB') > \mu, \forall \omega \text{ s.t. } |\omega| \geq \omega_{\max} + \delta.$$

Hence, (20) is satisfied with $|\omega| \geq \omega_{\max} + \delta$.

It remains to show (20) for $|\omega| < \omega_{\max} + \delta$. Note that $\psi - \omega\tau \in (-\frac{\pi}{2}, \frac{\pi}{2})$ by definition of δ and hence $\cos(\psi - \omega\tau) > 0$. Then

$$[A - \varepsilon\lambda e^{j\psi - j\omega\tau} BB']^* + [A - \lambda\varepsilon e^{j\psi - j\omega\tau} BB'] = -2\lambda\varepsilon\cos(\psi - \omega\tau) BB' \leq 0$$

Since (A, B) is controllable, we conclude that $A - \lambda\varepsilon e^{j\psi - j\omega\tau} BB'$ is Hurwitz, and hence (20) also holds, with $|\omega| < \omega_{\max} + \delta$. ∎

The next theorem addresses the consensus problem for networks with partial state coupling. In this case, a low-gain consensus controller can be designed as

$$\begin{cases} \dot{\chi}^i = (A + KC)\chi^i - Kz^i \\ u^i = \varepsilon B' \chi^i, \end{cases} \tag{21}$$

where K is such that $A + KC$ is Hurwitz.

Theorem 4 *For a given set* $\mathcal{G}_{0,\gamma,\varphi}$ *and* $\bar{\tau} > 0$*, consider the agents* (2) *with any communication topology belonging to* $\mathcal{G}_{0,\gamma,\varphi}$*. Suppose* $A + A' \leq 0$*. In that case, Problem 2 is solvable if,*

$$\bar{\tau}\omega_{\max} < \tfrac{\pi}{2} - \varphi \tag{22}$$

Moreover, it can be solved by the consensus controller (21) *if* (22) *holds. Specifically, for given* γ *and given* φ *and* $\bar{\tau}$ *satisfying* (22)*, there exists an* ε^* *such that for any* $\varepsilon \in (0, \varepsilon^*]$*, the agents* (2) *with controller* (21) *achieve consensus for any communication topology in* $\mathcal{G}_{0,\gamma,\varphi}$ *and* $\tau \in [0, \bar{\tau}]$.

Proof: It follows from Lemma A.2 in the Appendix that Theorem 2 holds if there exist $\alpha > 0$ and $\varepsilon^* > 0$ such that for $\varepsilon \in (0, \varepsilon^*]$, the system

$$\begin{cases} \dot{x}(t) = Ax(t) - \varepsilon\lambda e^{j\psi} BB' \chi(t-\tau) \\ \dot{\chi}(t) = (A + KC)\chi(t) - KCx(t) \end{cases} \tag{23}$$

is asymptotically stable for any $\lambda \in (0, \gamma)$, $\psi \in [-\varphi, \varphi]$ and $\tau \in [0, \bar{\tau}]$.

Define

$$\bar{\mathcal{A}} = \begin{bmatrix} A & 0 \\ -KC & A + KC \end{bmatrix}, \quad \bar{\mathcal{B}} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \bar{\mathcal{F}}_\varepsilon = \begin{bmatrix} 0 & -\varepsilon B' \end{bmatrix}.$$

By Lemma C.3 in the Appendix, there exists ε_1 such that for $\varepsilon \in (0, \varepsilon_1]$, $\bar{\mathcal{A}} + \alpha\lambda e^{j\psi} \bar{\mathcal{B}}\bar{\mathcal{F}}_\varepsilon$ is Hurwitz stable. It follows from Lemma B.1 that (23) is asymptotically stable if

$$\det\left[j\omega I - \bar{\mathcal{A}} - \alpha\lambda e^{j(\psi - \omega\tau)} \bar{\mathcal{B}}\bar{\mathcal{F}}_\varepsilon\right] \neq 0,\ \forall \omega \in \mathbb{R}, \quad \forall \lambda \in (\beta, \gamma),\ \forall \psi \in (-\varphi, \varphi),\ \forall \tau \in [0, \bar{\tau}]. \tag{24}$$

Similarly as before, there exist $\delta > 0$ and $\varepsilon_2 \leq \varepsilon_1$ such that for $\varepsilon \in (0, \varepsilon_2]$, condition (24) holds for $|\omega| \geq \omega_{\max} + \delta$. On the other hand, $|\omega| < \omega_{\max} + \delta$, it follows from Lemma C.3 that $\bar{\mathcal{A}} + \alpha\lambda e^{j(\psi - \omega\tau)} \bar{\mathcal{F}}_\varepsilon$ is Hurwitz stable. Therefore, condition (24) also holds with $|\omega| < \omega_{\max} + \delta$. ∎

6 Concluding remarks

In this paper, we study the multi-agent consensus with uniform constant communication delay for agents with high-order dynamics. A sufficient bound on the delay is derived under which the multi-agent consensus is attainable. Whenever this condition is satisfied, a controller without the exact knowledge of network topology can be constructed such that consensus can be achieved in a set of networks.

Although this paper focuses on unknown communication topologies, when the perfect information about the topology is in fact available, the design procedure can be easily modified to achieve a stronger result. In this case, input u_i to each agents can be first scaled as $u_i = d_i \bar{u}_i$ where these d_i are such that $\text{diag}\{d_i\}L$ has a simple eigenvalue at zero and the rest are real and strictly positive. The existence of such d_is is proved by Fisher & Fuller (1958). Then we can design $\bar{u}_i$ following the procedure proposed in this paper.

A Connection of network consensus to robust stabilization

The following lemmas are classical results in the study of multi-agent consensus problem (see Seo et al., 2009b, for instance).

Lemma A.1 *Problem 1 is solvable via consensus controller* $u^i = Fz^i$ *if the following* $N-1$ *systems*

$$\dot{\xi}^i(t) = A\xi^i(t) + \lambda_i BF\xi^i(t-\tau) \tag{A.1}$$

are asymptotically stable where λ_i, $i = 2, ..., N$ *are the non-zero eigenvalues of the Laplacian associated with the communication topology.*

Lemma A.2 *Problem 2 is solvable via consensus controller* (3) *if the following* $N-1$ *systems*

$$\begin{cases} \dot{x}^i(t) = Ax^i(t) + \lambda_i BC_c\chi^i(t-\tau) \\ \dot{\chi}^i(t) = A_c\chi^i(t) + B_c z^i(t) \end{cases} \tag{A.2}$$

are asymptotically stable where λ_i *for* $i = 2, \ldots, N$ *are the non-zero eigenvalues of the Laplacian matrix* L.

B Stability of linear time-delay system

The following lemma is adapted from Zhang et al. (2003).

Lemma B.1 *Consider a linear time-delay system*

$$\dot{x} = Ax + A_d x(t-\tau). \tag{B.1}$$

Assume $A + A_d$ *is Hurwitz. We have that* (B.1) *is globally asymptotically stable for* $\tau \in [0, \bar{\tau}]$ *if*

$$\det\left[j\omega I - A - e^{-j\omega\tau}A_d\right] \neq 0, \quad \forall \omega \in \mathbb{R},\ \forall \tau \in [0, \bar{\tau}],$$

for all $\omega \in \mathbb{R}$ *and* $\tau \in [0, \bar{\tau}]$.

C Robustness of low-gain state feedback and compensator

In this subsection, we recall some classical robust properties of low-gain feedback and compensator. Consider an uncertain system

$$\begin{cases} \dot{x} = Ax + \mu Bu \\ y = Cx, \end{cases} \tag{C.1}$$

where (A, B) is stabilizable, (A, C) is detectable and A has all its eigenvalues in the closed left half plane. The $\mu \in \mathbb{C}$ is input uncertainty. For $\varepsilon > 0$, let P_ε be the positive definite solution of ARE

$$A'P_\varepsilon + AP_\varepsilon - P_\varepsilon B'BP_\varepsilon + \varepsilon I = 0.$$

The robustness of a low-gain state feedback $u = -B'P_\varepsilon x$ is inherited from that of a classical LQR.

Lemma C.1 $A - \mu BB'P_\varepsilon$ *is Hurwitz stable for any* $\mu \in \{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq \frac{1}{2}\}$.

Proof: We observe that for $\mu \in \{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq \frac{1}{2}\}$,

$$\begin{aligned}(A - \mu BB'P_\varepsilon)^* P_\varepsilon &+ P_\varepsilon(A - \mu BB'P_\varepsilon) \\ &= -\varepsilon I - (2\,\mathrm{Re}(\mu) - 1)P_\varepsilon BB'P_\varepsilon < 0,\end{aligned}$$

and hence, $A - \mu BB'P_\varepsilon$ is Hurwitz stable. ■

The next lemma proves similar property of a low-gain compensator.

Lemma C.2 *For any a priori given bounded set*

$$\mathcal{W} \subseteq \{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq 1\},$$

there exists ε^* *such that for any* $\varepsilon \in (0, \varepsilon^*]$, *the closed-loop system of* (C.1) *and the low-gain compensator*

$$\begin{cases} \dot{\chi} = (A + KC)\chi - Ky, \\ u = -B'P_\varepsilon\chi \end{cases} \tag{C.2}$$

is asymptotically stable for any $\mu \in \mathcal{W}$.

Proof: Define $e = x - \chi$.The closed-loop of (C.1) and (C.2) can be rewritten in terms of x and e as

$$\begin{cases} \dot{x} = (A - \mu BB'P_\varepsilon)x + \mu BB'P_\varepsilon e \\ \dot{e} = (A + KC + \mu BB'P_\varepsilon)e - \mu BB'P_\varepsilon x. \end{cases} \tag{C.3}$$

Since $\mathrm{Re}(\mu) \geq 1$, we have

$$(A - \mu BB'P_\varepsilon)^* P_\varepsilon + P_\varepsilon(A - \mu BB'P_\varepsilon) \leq -\varepsilon I - P_\varepsilon BB'P_\varepsilon.$$

Define $V_1 = x^* P_\varepsilon x$ and $u = -B'P_\varepsilon x$. We can derive that

$$\dot{V}_1 \leq -\varepsilon\|x\|^2 - \|u\|^2 + \theta(\varepsilon)\|e\|\|u\|,$$

where $\theta(\varepsilon) = \|\mu B'P_\varepsilon\|$. Clearly, $\theta(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Let Q be the positive definite solution of Lyapunov equation

$$(A + KC)'Q + Q(A + KC) = -2I.$$

Since $F_\varepsilon \to 0$ and μ is bounded in $\mathcal{W}$, there exists ε_1 such that for $\varepsilon \in (0, \varepsilon_1]$,

$$(A + KC + \mu BB'P_\varepsilon)'Q + Q(A + KC + \mu BB'P_\varepsilon) \leq -I.$$

Define $V_2 = e^* Qe$. We get $\dot{V}_2 \leq -\|e\|^2 + M\|e\|\|u\|$ where $M = \max_{\mu \in \mathcal{W}}\{2\|\mu QB\|\}$.

Define $V = 4M^2 V_1 + 2V_2$. Then

$$\begin{aligned}\dot{V} \leq -4M^2\varepsilon\|x\|^2 2 - 2\|e\|^2 - 4M^2\|u\|^2 \\ + (4M^2\theta(\varepsilon) + 2M)\|e\|\|u\|\end{aligned}$$

There exist $\varepsilon^* \leq \varepsilon_1$ such that $4M^2\theta(\varepsilon) \leq 2M$ for $\varepsilon \in (0, \varepsilon^*]$. Hence for $\varepsilon \in (0, \varepsilon^*]$,

$$\dot{V} \leq -4M^2\varepsilon\|x\|^2 - \|e\|^2 - (\|e\| - 2M\|u\|)^2.$$

We conclude that (C.2) is asymptotically stable for any $\mu \in \mathcal{W}$. ■

Lemma C.3 *Consider system* (C.1). *Suppose $A' + A \leq 0$. For any a priori given $\varphi \in (0, \frac{\pi}{2})$ and a bounded set*

$$\mathcal{W} \subseteq \{s \in \mathbb{C} \mid s \neq 0, \quad \arg(s) \in [-\varphi, \varphi]\},$$

there exists ε^* such that for any $\varepsilon \in (0, \varepsilon^*]$, the closed-loop system of (C.1) *and the low-gain compensator*

$$\begin{cases} \dot{\chi} = (A + KC)\chi - Ky, \\ u = -\varepsilon B'\chi \end{cases} \tag{C.4}$$

is asymptotically stable for any $\mu \in \mathcal{W}$.

Proof: Define $e = x - \chi$. The closed-loop of (C.1) and (C.4) can be rewritten in terms of x and e as

$$\begin{cases} \dot{x} = (A - \varepsilon\mu BB')x + \varepsilon\mu BB'e \\ \dot{e} = (A + KC + \varepsilon\mu BB')e - \varepsilon\mu BB'x. \end{cases} \tag{C.5}$$

Define $V_1 = x^*x$ and $u = -B'x$. It is easy to get

$$\dot{V}_1 \leq -\varepsilon \operatorname{Re}(\mu)\|u\|^2 + \varepsilon|\mu|\theta_1\|e\|\|u\|,$$

where $\theta_1 = 2\|B\|$.

Let Q be the positive definite solution of Lyapunov equation

$$(A + KC)'Q + Q(A + KC) = -2I.$$

Since μ is bounded in $\mathcal{W}$, there exists ε_1 such that for $\varepsilon \in (0, \varepsilon_1]$,

$$(A + KC + \varepsilon\mu BB')^*Q + Q(A + KC + \varepsilon\mu BB') \leq -I.$$

Define $V_2 = e^*Qe$. We get $\dot{V}_2 \leq -\|e\|^2 + \varepsilon|\mu|\theta_2\|e\|\|u\|$ where $\theta_2 = 2\|QB\|$.

Define $V = V_1 + V_2$. Then with $\theta_3 = \theta_1 + \theta_2$, we can derive

$$\begin{aligned} \dot{V} \leq & -[1 - \varepsilon|\mu|\sec(\varphi)]\,\|e\|^2 - \frac{3}{4}\varepsilon \operatorname{Re}(\mu)\|u\|^2 \\ & - \varepsilon \operatorname{Re}(\mu)\left(\tfrac{1}{2}\|u\| - \sec(\varphi)\theta_3\|e\|\right)^2 \end{aligned}$$

Since $\mathcal{W}$ is bounded and φ is given, there exists $\varepsilon^* \leq \varepsilon_1$ such that for $\varepsilon \in (0, \varepsilon^*]$, $\varepsilon|\mu|\sec(\varphi) \leq \frac{1}{2}$, $\forall \mu \in \mathcal{W}$. Hence for $\varepsilon \in (0, \varepsilon^*]$,

$$\dot{V} \leq -\frac{1}{2}\|e\|^2 - \frac{3}{4}\varepsilon \operatorname{Re}(\mu)\|u\|^2.$$

Since (A, B) is stabilizable, we conclude that (C.2) is asymptotically stable for any $\mu \in \mathcal{W}$. ■

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