Parameter Estimation and Compensation in Systems with Nonlinearly Parameterized Perturbations ⋆

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Abstract

We consider a class of systems influenced by perturbations that are nonlinearly parameterized by unknown constant parameters, and develop a method for estimating the unknown parameters. The method applies to systems where the states are available for measurement, and perturbations with the property that an exponentially stable estimate of the unknown parameters can be obtained if the whole perturbation is known. The main contribution is to introduce a conceptually simple, modular design that gives freedom to the designer in accomplishing the main task, which is to construct an update law to asymptotically invert a nonlinear equation. Compensation for the perturbations in the system equations is considered for a class of systems with uniformly globally bounded solutions, for which the origin is uniformly globally asymptotically stable when no perturbations are present. We also consider the case when the parameters can only be estimated when the controlled state is bounded away from the origin, and show that we may still be able to achieve convergence of the controlled state. We illustrate the method through examples, and apply it to the problem of downhole pressure estimation during oil well drilling.

Key words: Nonlinear system control; Uncertain nonlinear systems; Adaptive control; Nonlinear observer design

1 Introduction

An important issue in model-based control is the handling of unknown perturbations to system equations. Such perturbations can be the result of external disturbances or internal plant changes, such as a configuration change, system fault, or changes in physical plant characteristics. Frequently, the perturbations can be characterized in terms of a vector of unknown, constant parameters.

Adaptive control techniques counteract such perturbations by using estimates of the unknown parameters that are updated online. When the perturbations are linear with respect to the unknown parameters, adaptive control design is often straightforward, and techniques for handling such cases are well developed (see, e.g., Krstić, Kanellakopoulos, and Kokotović, 1995; Ioannou and Sun, 1996). In the nonlinear case the range of available design techniques is more limited. One approach is to use a gradient algorithm, as in linearly parameterized systems, which may yield poor results or unstable behavior for nonlinear parameterizations (see discussion in Annaswamy, Skantze, and Loh, 1998). Another common strategy is implementing an extended Kalman filter (EKF) for estimation of the unknown parameters. Although this often yields good results, analysis of the stability properties of an EKF is difficult (see, e.g., Reif, Günther, Yaz, and Unbehauen, 1999). Introducing extra parameters to obtain a linear expression is sometimes possible, but doing so may increase the complexity and affect the performance by reducing the convergence rate of the parameter estimates or introducing stricter persistency-of-excitation conditions.

Some techniques that do not resort to approximations are found in the literature. In Fomin, Fradkov, and Yakubovich (1981); Ortega (1996), the stability and convergence of the controlled state are proven for a gradient-type approach for nonlinear parameterizations with a convexity property. Annaswamy et al. (1998) exploit the convexity or concavity of some parameterizations by introducing a tuning function and adaptation based on a min-max optimization strategy, achieving arbitrarily accurate tracking of the controlled states. This approach is extended to general nonlinear parameterizations in Loh, Annaswamy, and Skantze (1999), and parameter convergence is studied in Cao, Annaswamy, and Kojić (2003). Other results, such as Bošković (1995, 1998); Zhang, Ge, Hang, and Chai (2000), focus on first-order systems with certain fractional parameterizations, proving con-

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vergence of the controlled state. In Qu (2003), an approach is introduced for a class of higher-order systems with matrix fractional parameterizations, which achieves global boundedness and ultimate boundedness with a desired precision. Here, an auxiliary estimate of the full perturbation is used in the estimation of the unknown parameters, the same principle upon which the results in this paper are based. In Qu, Hull, and Wang (2006), an approach for more general nonlinear parameterizations is presented, where the parameter estimate used in the control law is biased by an appropriately chosen vector function.

Another way of dealing with undesired perturbations is found in Chakrabortty and Arcak (2009), where a high-gain approach is used to produce an estimate of the perturbation, which is then used for control. By increasing the gain, the estimate is made to converge arbitrarily fast, and the transient performance of the unperturbed system can therefore be recovered. The approach considered in this paper has similarities to Chakrabortty and Arcak (2009), but it also exploits available structural information by estimating an unknown parameter vector in addition to the full perturbation. The parameter estimate is produced by a parameter estimation module that is designed as if the perturbation were known. In the actual implementation, however, the estimate of the perturbation is used. This idea is similar to the ideas in Tyukin (2003), where adaptive update laws of a certain structure, called *virtual algorithms*, are designed as if time derivatives of the measurements were available, before being transformed into realizable form without explicit differentiation of the measurements. In Tyukin, Prokhorov, and van Leeuwen (2007), this principle is used to design a family of adaptation laws for monotonically parameterized perturbations in the first derivatives.

The main contribution of this paper is a nonlinear parameter estimation design with a clear modular structure. The design is split into two modules: a perturbation estimator, and a parameter estimator constructed by the designer to asymptotically invert a nonlinear equation. The modular structure is conceptually simple, and it isolates the task of inverting the nonlinear equation, giving the designer freedom in how to best accomplish this task. Through a series of propositions, we provide guidelines for how to construct the parameter estimator, and we obtain explicit Lyapunov functions that prove exponential convergence of the parameter estimates. One of the main advantages of the modular structure is extensibility. In particular, it is demonstrated in Grip, Saberi, and Johansen (2009) that the perturbation estimator can be extended to handle systems with partial state measurement, where the perturbation appears in higher-order derivatives of the output. The method can often be used to provide fast parameter estimates, which may be useful not only for direct compensation, but as part of other control schemes where fast parameter estimates are required, for example, traditional adaptive approaches combined with parameter resetting (see, e.g., Bakkeheim, Johansen, Smogeli, and Sørensen, 2008).

1.1 Preliminaries

We use conventional notation to denote estimates and error variables. For a vector z, $\hat{z}$ represents its estimate and $\tilde{z} = z - \hat{z}$ is an error variable. We denote by z_i the i'th element of z, when this is clear from the context. The norm operator $\|\cdot\|$ denotes the Euclidean norm for vectors and the induced Euclidean norm for matrices. For a symmetric, positive-definite matrix P and a vector z, we write $\|z\|_P = (z^\mathsf{T} P z)^{1/2}$. The maximum and minimum eigenvalues of a symmetric matrix P are denoted $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$. The open and closed balls around the origin with radius ε are denoted $B(\varepsilon)$ and $\overline{B}(\varepsilon)$, respectively. We denote by $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{>0}$ the nonnegative and the positive real numbers. For a set $E \subset \mathbb{R}^n$, we write $(E - E) := \{z_1 - z_2 \in \mathbb{R}^n \mid z_1, z_2 \in E\}$. Throughout this paper, when considering systems of the form $\dot{z} = F(t,z)$, we implicitly assume that $F\colon \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$ is piecewise continuous in t and locally Lipschitz continuous in z, uniformly in t, on $\mathbb{R}_{\geq 0} \times \mathbb{R}^n$. The solution of this system, initialized at time $t_0 \geq 0$ with initial condition $z(t_0)$, is denoted $z(t)$. The continuity assumptions on F allow us to conclude that $z(t)$ is uniquely defined for all $t \geq t_0$ if it is known not to escape from some compact subset of the state space (see Khalil, 2001, Ch. 3).

2 Problem Formulation

We consider systems that, by the appropriate state transformations and choice of control law, can be expressed in the following form:

$$\dot{x} = f(t,x) + B(t,x)\left(g(t,x,\theta) + v(t,x)\right), \qquad (1)$$

where $x \in \mathbb{R}^n$ is a measured state vector and $\theta \in \mathbb{R}^p$ is a vector of unknown, constant parameters. The functions $f\colon \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$, $B\colon \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ and $v\colon \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^m$ can be evaluated from available measurements, and $g\colon \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^m$ can be evaluated if θ is known.

In most practical circumstances, it is known from physical considerations that θ is restricted to some bounded set of values. This is a significant advantage when it comes to satisfying the assumptions made later in this paper. To simplify the exposition, we therefore assume that the set of possible parameters is bounded. In designing update laws for parameter estimates, we also assume that a parameter projection can be implemented as described in Krstić et al. (1995), restricting the parameter estimates to a compact, convex set $\Theta \subset \mathbb{R}^p$, defined slightly larger than the set of possible parameter values. The parameter projection is denoted $\mathrm{Proj}(\cdot)$, and is described in Appendix A.

All functions on the right-hand side of (1) are well defined and bounded for each bounded $(t,x,\theta) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \Theta$. We assume that $g(t,x,\theta)$ is continuously differentiable with respect to θ for all $\theta \in \Theta$. As we construct an estimator, additional smoothness requirements may be needed for f,

B, g, and v, as well as other functions, to guarantee that the piecewise continuity and local Lipschitz conditions in Section 1.1 hold for all systems involved. It is left to the designer to check these requirements.

3 Parameter Estimation

In this section, we present a method for estimating the unknown parameter vector θ when $x(t)$ is bounded. Let $\phi := B(t,x)g(t,x,\theta)$ represent the full unknown perturbation in (1). The idea behind the estimation scheme is as follows. We first design an update law that exponentially estimates θ based on the quantity ϕ, as though ϕ were known. We then produce an estimate of ϕ and implement the update law based on this estimate instead of the real perturbation.

3.1 Estimation of θ from ϕ

We denote by $\hat{\phi}$ the estimate of the perturbation ϕ. We shall later explain how to construct this estimate; for now, we concentrate on how to find θ in the hypothetical case of a perfect perturbation estimate. For this to work, there needs to exist an update law

$$\dot{\hat{\theta}} = u_\theta(t,x,\hat{\phi},\hat{\theta}), \tag{2}$$

which, if $\hat{\phi} = \phi$, would provide an unbiased asymptotic estimate of θ. This is the subject of the following assumption on the dynamics of the error variable $\tilde{\theta} := \theta - \hat{\theta}$.

Assumption 1 *For each compact set $K \subset \mathbb{R}^n$, there exist a differentiable function $V_\mathrm{u}\colon \mathbb{R}_{\geq 0} \times (\Theta - \Theta) \to \mathbb{R}_{\geq 0}$; positive constants a_1, a_2 and a_4; and a continuous function $a_3\colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ that is positive outside the origin, such that, for all $(t,x,\phi,\hat{\theta}) \in \mathbb{R}_{\geq 0} \times K \times \mathbb{R}^n \times \Theta$,*

$$a_1\|\tilde{\theta}\|^2 \leq V_\mathrm{u}(t,\tilde{\theta}) \leq a_2\|\tilde{\theta}\|^2, \tag{3}$$

$$\frac{\partial V_\mathrm{u}}{\partial t}(t,\tilde{\theta}) - \frac{\partial V_\mathrm{u}}{\partial \tilde{\theta}}(t,\tilde{\theta}) u_\theta(t,x,\phi,\hat{\theta}) \leq -a_3(x)\|\tilde{\theta}\|^2, \tag{4}$$

$$\left\| \frac{\partial V_\mathrm{u}}{\partial \tilde{\theta}}(t,\tilde{\theta}) \right\| \leq a_4\|\tilde{\theta}\|. \tag{5}$$

Furthermore, the update law (2) *ensures that, if $\hat{\theta}(t_0) \in \Theta$, then, for all $t \geq t_0$, $\hat{\theta}(t) \in \Theta$.*

Satisfying Assumption 1 constitutes the greatest challenge in applying the method in this paper, and this is therefore discussed in detail in the next section.

3.2 Satisfying Assumption 1

We shall focus mostly on the case when Assumption 1 can be satisfied with $a_3(x) \geq a_3^* > 0$. This guarantees that the origin of the error dynamics $\dot{\tilde{\theta}} = -u_\theta(t,x,\phi,\theta - \tilde{\theta})$, which occurs if $\hat{\phi} = \phi$, is uniformly exponentially stable with $(\Theta - \Theta)$ contained in the region of attraction. Essentially this amounts to asymptotically solving the inversion problem of finding θ given $\phi = B(t,x)g(t,x,\theta)$. In the following, we shall discuss some possibilities for how to satisfy Assumption 1. As a useful reference, we point to Nicosia, Tornambè, and Valigi (1994), which deals with the use of state observers for inversion of nonlinear maps.

The most obvious way to satisfy Assumption 1 is to invert the equality $\phi = B(t,x)g(t,x,\theta)$ algebraically, and to let $\hat{\theta}$ be attracted to this solution.

Proposition 1 *Suppose that, for all $(t,x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$, we can find a unique solution for θ from the equation $\phi = B(t,x)g(t,x,\theta)$. Then Assumption 1 is satisfied with $a_3(x) \geq a_3^* > 0$ by using the update law $u_\theta(t,x,\hat{\phi},\hat{\theta}) = \mathrm{Proj}(\Gamma(\theta^*(t,x,\hat{\phi}) - \hat{\theta}))$, where $\theta^*(t,x,\hat{\phi})$ denotes the solution of the inversion problem found from $\hat{\phi}$, and Γ is a symmetric positive-definite gain matrix.* □

PROOF The proof follows from using the function $V_\mathrm{u}(t,\tilde{\theta}) = \frac{1}{2}\tilde{\theta}^\intercal\Gamma^{-1}\tilde{\theta}$ when $\hat{\phi} = \phi$, and the property (Krstić et al., 1995, Lemma E.1) that $-\tilde{\theta}^\intercal\Gamma^{-1}\mathrm{Proj}(\tau) \leq -\tilde{\theta}^\intercal\Gamma^{-1}\tau$. From the proof of Krstić et al. (1995, Lemma E.1), we find that, if $\hat{\theta}(t_0) \in \Theta$, then the solution $\hat{\theta}(t)$ remains in Θ. ■

Example 1 Consider the perturbation $B(t,x)g(t,x,\theta) = h((2+\sin(t))\theta)$, where h is some explicitly invertible, nonlinear mapping. For each $t \in \mathbb{R}_{\geq 0}$, we can solve the inversion problem and find $\theta^*(t,x,\hat{\phi}) = h^{-1}(\hat{\phi})/(2+\sin(t))$. □

Often it is only possible to invert the equation $\phi = B(t,x)g(t,x,\theta)$ part of the time. In this case, Assumption 1 may still be satisfied if solutions are available with a certain regularity. The following proposition deals with this case.

Proposition 2 *Suppose that there exists a known, piecewise continuous function $l\colon \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to [0,1]$, and that, for all $(t,x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$, $l(t,x) > 0$ implies that we can find a unique solution for θ from the equation $\phi = B(t,x)g(t,x,\theta)$. Suppose furthermore that there exist $T > 0$ and $\varepsilon > 0$ such that, for all $t \in \mathbb{R}_{\geq 0}$, $\int_t^{t+T} l(\tau,x(\tau))\,\mathrm{d}\tau \geq \varepsilon$. Then Assumption 1 is satisfied with $a_3(x) \geq a_3^* > 0$ by using the update law $u_\theta(t,x,\hat{\phi},\hat{\theta}) = \mathrm{Proj}(l(t,x)\Gamma(\theta^*(t,x,\hat{\phi}) - \hat{\theta}))$, where $\theta^*(t,x,\hat{\phi})$ denotes the solution of the inversion problem found from $\hat{\phi}$ whenever $l(t,x) > 0$, and Γ is a symmetric positive-definite gain matrix.* □

PROOF See Appendix B. ■

Example 2 Consider the perturbation $B(t,x)g(t,x,\theta) = h(\sin(t)\theta)$, where h is some explicitly invertible, nonlinear mapping. The inversion problem is poorly conditioned when $\sin(t)$ is close to zero, and unsolvable for $\sin(t) = 0$. Proposition 2 nevertheless applies by letting, for example, $l(t,x) = 0$ when $|\sin(t)| < \varepsilon$ and $l(t,x) = 1$ when $|\sin(t)| \geq \varepsilon$, where $0 < \varepsilon < 1$. □

When it is not possible or desirable to solve the inversion problem explicitly, it is often possible to implement the update function as a numerical search for the solutions.

Proposition 3 *Suppose that there exist a symmetric positive-definite matrix P and a function $M\colon \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \Theta \to \mathbb{R}^{p\times n}$ such that, for all $(t,x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$, and for all pairs $\theta_1, \theta_2 \in \Theta$,*

$$M(t,x,\theta_1)B(t,x)\frac{\partial g}{\partial \theta}(t,x,\theta_2) + \frac{\partial g}{\partial \theta}^{\mathsf{T}}(t,x,\theta_2)B^{\mathsf{T}}(t,x)M^{\mathsf{T}}(t,x,\theta_1) \geq 2P. \quad (6)$$

Then Assumption 1 is satisfied with $a_3(x) \geq a_3^* > 0$ by using the update law $u_\theta(t,x,\hat{\phi},\hat{\theta}) = \mathrm{Proj}(\Gamma M(t,x,\hat{\theta})(\hat{\phi} - B(t,x)g(t,x,\hat{\theta})))$, where Γ is a symmetric positive-definite gain matrix. □

PROOF See Appendix B. ■

Example 3 Consider the perturbation $B(t,x)g(t,x,\theta) = g(\theta) = [\theta_1,\, \theta_1^2 + \theta_2]^{\mathsf{T}}$. Selecting $M(t,x,\hat{\theta}) = M = \mathrm{diag}(K_M, 1)$ yields $M[\partial g/\partial\theta](\theta) + [\partial g/\partial\theta]^{\mathsf{T}}(\theta)M^{\mathsf{T}} = 2\left[\begin{smallmatrix} K_M & \theta_1 \\ \theta_1 & 1\end{smallmatrix}\right]$. Using the fact that θ_1 is bounded within Θ, it is easily confirmed that, if K_M is chosen sufficiently large, then $M[\partial g/\partial\theta](\theta) + [\partial g/\partial\theta]^{\mathsf{T}}(\theta)M^{\mathsf{T}} \geq 2P$, where P is symmetric positive definite. □

Proposition 3 applies to certain monotonic perturbations for which a solution can be found arbitrarily fast by increasing the gain Γ. In many cases, this is not possible, because the inversion problem is singular the whole time or part of the time. The following proposition applies to cases where a solution is only available by using data over longer periods of time, by incorporating a persistency-of-excitation condition.

Proposition 4 *Suppose that there exist a piecewise continuous function $S\colon \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{S}_{p+}$, where $\mathbb{S}_{p+}$ is the cone of $p \times p$ symmetric positive-semidefinite matrices, and a function $M\colon \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \Theta \to \mathbb{R}^{p\times n}$, both bounded for bounded x, such that, for all $(t,x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$, and for all pairs $\theta_1, \theta_2 \in \Theta$,*

$$M(t,x,\theta_1)B(t,x)\frac{\partial g}{\partial \theta}(t,x,\theta_2) + \frac{\partial g}{\partial \theta}^{\mathsf{T}}(t,x,\theta_2)B^{\mathsf{T}}(t,x)M^{\mathsf{T}}(t,x,\theta_1) \geq 2S(t,x). \quad (7)$$

Suppose furthermore that there exist numbers $T > 0$ and $\varepsilon > 0$ such that, for all $t \in \mathbb{R}_{\geq 0}$, $\int_t^{t+T} S(\tau, x(\tau))\,\mathrm{d}\tau \geq \varepsilon I$, and that, for all $(t,x,\hat{\theta}) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \Theta$, $\|B(t,x)(g(t,x,\theta) - g(t,x,\hat{\theta}))\| \leq L_g(\tilde{\theta}^{\mathsf{T}}S(t,x)\tilde{\theta})^{1/2}$, for some $L_g > 0$. Then Assumption 1 is satisfied with $a_3(x) \geq a_3^* > 0$ by using the update law $u_\theta(t,x,\hat{\phi},\hat{\theta}) = \mathrm{Proj}(\Gamma M(t,x,\hat{\theta})(\hat{\phi} - B(t,x)g(t,x,\hat{\theta})))$, where Γ is a symmetric positive-definite gain matrix. □

PROOF See Appendix B. ■

Example 4 Consider the perturbation from Example 3 multiplied by $\sin(t)$; that is, $B(t,x)g(t,x,\theta) = g(t,\theta) = \sin(t)[\theta_1,\, \theta_1^2 + \theta_2]^{\mathsf{T}}$. Using the same argument as in Example 3, we may choose $M(t,x,\hat{\theta}) = M(t) = \sin(t)\mathrm{diag}(K_M, 1)$ to satisfy (7). We then have $S(t,x) = S(t) = \sin^2(t)P$, where P is the positive-definite matrix from Example 3. For any $T > 0$ there is an $\varepsilon > 0$ such that $\int_t^{t+T} P\sin^2(\tau)\,\mathrm{d}\tau \geq \varepsilon I$ for all $t \in \mathbb{R}_{\geq 0}$, which means that the integral condition in Proposition 4 is satisfied. Finally, we have $\|g(t,\theta) - g(t,\hat{\theta})\| \leq L_g(\tilde{\theta}^{\mathsf{T}}S(t)\tilde{\theta})^{1/2}$, where $L_g = \max_{(t,\theta)\in\mathbb{R}_{\geq 0}\times\Theta} \|[\partial g/\partial\theta](t,\theta)\|/\lambda_{\min}(P)^{1/2}$. Hence, Proposition 4 applies. □

Remark 1 When looking for the function M, a good starting point is $M(t,x,\hat{\theta}) = [\partial g/\partial\theta]^{\mathsf{T}}(t,x,\hat{\theta})B^{\mathsf{T}}(t,x)$. This choice makes the parameter update law into a gradient search in the direction of steepest descent for the function $\|B(t,x)(g(t,x,\theta) - g(t,x,\hat{\theta}))\|^2$, scaled by the gain Γ. Indeed, this choice of M often works even if it fails to satisfy either of Propositions 3 or 4. In the special case where the perturbation is linear in the unknown parameters, this choice of M always satisfies (7), and the remaining conditions in Proposition 4 coincide with standard persistency-of-excitation conditions for parameter identification in linear adaptive theory (see, e.g., Marino and Tomei, 1995, Ch. 5). Future research will include investigation of more systematic ways of finding the function M for nonlinear parameterizations. □

We end this section with an example illustrating that the above approaches may be combined.

Example 5 Consider the perturbation $B(t,x)g(t,x,\theta) = [\theta_1^{1/3},\, \sin(\theta_1 a(t))\theta_2]^{\mathsf{T}}$ with θ known to be bounded and θ_1 known to be bounded away from zero, and where $a(t)$ is some persistently excited signal with a bounded derivative. Clearly, we can find θ_1 by inversion, simply taking $\theta_1^*(\hat{\phi}) = \hat{\phi}_1^3$. Hence, θ_1 is handled according to Proposition 1. When θ_1 is known, we can find θ_2 by numerical search according to Proposition 4. We therefore implement the second part of the update law according to Proposition 4, substituting θ_1 with $\hat{\phi}^3$, which results in $u_\theta(t,x,\hat{\phi},\hat{\theta}) = \mathrm{Proj}(\Gamma[\hat{\phi}_1^3 - \hat{\theta}_1,\, \sin(\hat{\phi}_1^3 a(t))(\hat{\phi}_2 - \sin(\hat{\phi}_1^3 a(t))\hat{\theta}_2)]^{\mathsf{T}})$. □

3.3 Estimator

We now introduce the full estimator:

$$\dot{z} = -K_\phi\big(f(t,x) + B(t,x)v(t,x) + \hat{\phi}\big) - B(t,x)\frac{\partial g}{\partial\theta}(t,x,\hat{\theta})u_\theta(t,x,\hat{\phi},\hat{\theta}), \quad (8a)$$

$$\hat{\phi} = z + K_\phi x + B(t,x)g(t,x,\hat{\theta}), \quad (8b)$$

$$\dot{\hat{\theta}} = u_\theta(t,x,\hat{\phi},\hat{\theta}), \quad (8c)$$

where K_ϕ is a symmetric positive-definite gain matrix. The full estimator consists of two parts: an estimator for ϕ, described by (8a), (8b), and the update law from Section 3.1. To study the properties of the estimator, we consider the dynamics of the errors $\tilde{\phi}$ and $\tilde{\theta}$. Taking the time derivative of $\tilde{\phi} = \phi - \hat{\phi}$, we may write

$$\begin{aligned}\dot{\tilde{\phi}} &= K_\phi\big(f(t,x)+B(t,x)v(t,x)+\hat{\phi}\big)\\ &\quad + B(t,x)\frac{\partial g}{\partial \theta}(t,x,\hat{\theta})u_\theta(t,x,\hat{\phi},\hat{\theta}) - K_\phi \dot{x}\\ &\quad - B(t,x)\frac{\partial g}{\partial \theta}(t,x,\hat{\theta})u_\theta(t,x,\hat{\phi},\hat{\theta}) + d(t,x,\tilde{\theta}),\end{aligned} \tag{9}$$

where

$$\begin{aligned}d(t,x,\tilde{\theta}) &:= \frac{\partial}{\partial t}\big(B(t,x)(g(t,x,\theta)-g(t,x,\hat{\theta}))\big)\\ &\quad + \frac{\partial}{\partial x}\big(B(t,x)(g(t,x,\theta)-g(t,x,\hat{\theta}))\big)\dot{x}.\end{aligned} \tag{10}$$

The function $d(t,x,\tilde{\theta})$ can be seen as the time derivative of $B(t,x)(g(t,x,\theta)-g(t,x,\hat{\theta}))$ when $\hat{\theta}$ is kept constant. Using the expression $\dot{x} - f(t,x) - B(t,x)v(t,x) = \phi$, we may rewrite the above expression and write the error dynamics of the estimator as

$$\dot{\tilde{\phi}} = -K_\phi\tilde{\phi} + d(t,x,\tilde{\theta}), \tag{11a}$$

$$\dot{\tilde{\theta}} = -u_\theta(t,x,\phi,\hat{\theta}) + \big(u_\theta(t,x,\phi,\hat{\theta}) - u_\theta(t,x,\hat{\phi},\hat{\theta})\big). \tag{11b}$$

For convenience, we define the error variable $\xi := [\tilde{\phi}^\top, \tilde{\theta}^\top]^\top$ and the set $\Xi := \mathbb{R}^n \times (\Theta - \Theta)$.

Assumption 2 *For all $(t,x,\tilde{\theta}) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times (\Theta-\Theta)$, the function $d(t,x,\tilde{\theta})$ is well defined; for each compact set $K \subset \mathbb{R}^n$, there exist continuous functions $L_1(x) > 0$ and $L_2(x) > 0$ such that, for all $(t,x,\tilde{\theta}) \in \mathbb{R}_{\geq 0} \times K \times (\Theta-\Theta)$, $\|d(t,x,\tilde{\theta})\| \leq L_1(x)\|\tilde{\theta}\|$; and for all $(t,x,\phi,\hat{\phi},\hat{\theta}) \in \mathbb{R}_{\geq 0} \times K \times \mathbb{R}^n \times \mathbb{R}^n \times \Theta$, $\|u_\theta(t,x,\phi,\hat{\theta}) - u_\theta(t,x,\hat{\phi},\hat{\theta})\| \leq L_2(x)\|\tilde{\phi}\|$.*

Remark 2 When checking the condition $\|u_\theta(t,x,\phi,\hat{\theta}) - u_\theta(t,x,\hat{\phi},\hat{\theta})\| \leq L_2(x)\|\tilde{\phi}\|$, the projection in the update law can be disregarded, because the property is retained under projection (see Appendix A.1). ▫

The Lipschitz-type conditions in Assumption 2 may appear difficult to satisfy. Note, however, that $\tilde{\theta} \in (\Theta - \Theta)$, which means that we are dealing with a local Lipschitz condition for d. For u_θ, we need to satisfy a global condition in the sense that ϕ and $\hat{\phi}$ are not presumed bounded. Indeed, such a condition may often fail to hold, as demonstrated by Example 5, where the term $\hat{\phi}_1^3$ is used. In most cases, however, the perturbation ϕ depends on physical quantities with known bounds, and from these a bound on ϕ can often be found. It is then possible to modify u_θ to include a saturation of $\hat{\phi}$, thereby reducing the requirement to a local condition that is much more easily satisfied. With the inclusion of a saturation, Example 5 does satisfy Assumption 2. If a particular update law is modified by including a saturation, it does not affect the validity of conditions (3)–(5) in Assumption 1, since the saturation has no effect when $\hat{\phi} = \phi$.

Theorem 1 *Suppose that Assumptions 1 and 2 hold with $a_3(x) \geq a_3^* > 0$ and that, for all $t \in \mathbb{R}_{\geq 0}$, $\|x(t)\|$ is uniformly bounded. Then there exists $k_\phi > 0$ such that, if K_ϕ is chosen such that $\lambda_{\min}(K_\phi) > k_\phi$, then the origin of (11) is uniformly exponentially stable with Ξ contained in the region of attraction.* ▫

PROOF By assumption, $x(t) \in K$, where $K \subset \mathbb{R}^n$ is a compact set. We can therefore make use of Assumptions 1 and 2 for this particular K. Boundedness of x ensures that ϕ is well defined for all times. By Assumption 1, $\hat{\theta}(t_0) \in \Theta$ implies that the solution $\hat{\theta}(t)$ cannot escape Θ, and hence $\tilde{\theta}(t)$ cannot escape $(\Theta - \Theta)$. From (11a) and the expression $\|d(t,x,\tilde{\theta})\| \leq L_1^*\|\tilde{\theta}\|$, where L_1^* is a bound on $L_1(x)$ on K, it is therefore easily seen that $\tilde{\phi}$ remains bounded. Thus, if $\xi(t_0) \in \Xi$, then, for all $t \geq t_0$, $\xi(t)$ is uniquely defined and remains in a compact subset of Ξ. We define the function $V_{\mathrm{p}}(t,\xi) = V_{\mathrm{u}}(t,\tilde{\theta}) + \frac{1}{2}\tilde{\phi}^\top\tilde{\phi}$ and investigate its time derivative on the set Ξ along the trajectories of (11):

$$\begin{aligned}\dot{V}_{\mathrm{p}}(t,\xi) &= \frac{\partial V_{\mathrm{u}}}{\partial t}(t,\tilde{\theta}) - \frac{\partial V_{\mathrm{u}}}{\partial \tilde{\theta}}(t,\tilde{\theta})u_\theta(t,x,\phi,\hat{\theta})\\ &\quad + \frac{\partial V_{\mathrm{u}}}{\partial \tilde{\theta}}(t,\tilde{\theta})\big(u_\theta(t,x,\phi,\hat{\theta}) - u_\theta(t,x,\hat{\phi},\hat{\theta})\big)\\ &\quad - \tilde{\phi}^\top K_\phi\tilde{\phi} + \tilde{\phi}^\top d(t,x,\tilde{\theta}).\end{aligned} \tag{12}$$

From the inequalities in Assumptions 1 and 2,

$$\begin{aligned}\dot{V}_{\mathrm{p}}(t,\xi) &\leq -a_3(x)\|\tilde{\theta}\|^2 - \lambda_{\min}(K_\phi)\|\tilde{\phi}\|^2 + \|\tilde{\phi}\|\|d(t,x,\tilde{\theta})\|\\ &\quad + \left\|\frac{\partial V_{\mathrm{u}}}{\partial \tilde{\theta}}(t,\tilde{\theta})\right\|\|u_\theta(t,x,\phi,\hat{\theta}) - u_\theta(t,x,\hat{\phi},\hat{\theta})\|.\end{aligned} \tag{13}$$

This expression can be rewritten as $\dot{V}_{\mathrm{p}}(t,\xi) \leq -\zeta^\top Q(x)\zeta$, where $\zeta = [\|\tilde{\phi}\|, \|\tilde{\theta}\|]^\top$ and

$$Q(x) = \begin{bmatrix} \lambda_{\min}(K_\phi) & -\frac{1}{2}(a_4L_2(x)+L_1(x)) \\ -\frac{1}{2}(a_4L_2(x)+L_1(x)) & a_3(x)\end{bmatrix}. \tag{14}$$

To check for positive-definiteness of $Q(x)$, we note that its first-order leading principal minor is $\lambda_{\min}(K_\phi) > 0$. The second-order leading principal minor is $a_3(x)\lambda_{\min}(K_\phi) - \frac{1}{4}(a_4L_2(x) + L_1(x))^2$, which is positive if $\lambda_{\min}(K_\phi) > k_\phi := (a_4L_2^* + L_1^*)^2/(4a_3^*)$, where L_2^* is a bound on $L_2(x)$ on K. Hence, we have on Ξ that $\dot{V}_{\mathrm{p}}(t,\xi(t)) \leq -\lambda_{\min}(Q(x))\|\xi(t)\|^2$. Moreover, we have that $V_{\mathrm{p}}(t,\xi) \leq \max\{a_2, \frac{1}{2}\}\|\xi\|^2$. From the preceding two expressions, we have that $\dot{V}_{\mathrm{p}}(t,\xi(t)) \leq -2\lambda V_{\mathrm{p}}(t,\xi(t))$, where $\lambda := \min_{x\in K}\lambda_{\min}(Q(x))/\max\{2a_2,1\}$. By the comparison lemma (Khalil, 2001, Lemma 3.4), we therefore have $V_{\mathrm{p}}(t,\xi(t)) \leq V_{\mathrm{p}}(t_0,\xi(t_0))\exp(-2\lambda(t-t_0))$. This leads to $\|\xi(t)\| \leq k_e\|\xi(t_0)\|\exp(-\lambda(t-t_0))$, where $k_e = (\max\{a_2, \frac{1}{2}\}/\min\{a_1, \frac{1}{2}\})^{1/2}$. ■

Remark 3 We assume in Theorem 1 that the state x is uniformly bounded. In pure estimation problems, where no control is implemented based on the parameter estimates, this is usually a reasonable assumption, because the states involved are typically derived from bounded physical quantities. □

4 Closed-Loop Compensation

We now consider how the parameter estimates can be used to compensate for the perturbation in (1). Suppose that the control inputs available in the original system can be chosen to yield a system on the following form:

$$\dot{x} = f(t,x) + B(t,x)\big(g(t,x,\theta) - g(t,x,\hat{\theta})\big). \tag{15}$$

Here, $v(t,x)$ in (1) has been substituted with the control $-g(t,x,\hat{\theta})$.

Assumption 3 *The function $f(t,x)$ is continuously differentiable on $\mathbb{R}_{\geq 0} \times \mathbb{R}^n$; the origin of the nominal system $\dot{x} = f(t,x)$ is uniformly globally asymptotically stable (UGAS); for any trajectory $\hat{\theta}(t) \in \Theta$, the solutions $x(t)$ of the perturbed system (15) are uniformly globally bounded (UGB); and for each compact set $K \subset \mathbb{R}^n$ there exists a class $\mathcal{K}$ function γ such that, for all $(t,x,\hat{\theta}) \in \mathbb{R}_{\geq 0} \times K \times \Theta$, $\|B(t,x)(g(t,x,\theta) - g(t,x,\hat{\theta}))\| \leq \gamma(\|\tilde{\theta}\|)$.*

In Assumption 3, we assume that $f(t,x)$ is a stabilizing function that ensures UGB irrespective of the parameter estimate. In this case, the only control needed is a term $-g(t,x,\hat{\theta})$ to cancel the perturbation, as seen in (15). Essentially, the UGB assumption means that a parameter error confined to $(\Theta - \Theta)$ cannot make the states of the system arbitrarily large compared to their initial values. In many cases, this assumption is not automatically satisfied, and one may need to apply additional control to shape $f(t,x)$. The assumption is most easily satisfied if the asymptotic growth rate of $f(t,x)$ with respect to x is greater than the asymptotic growth rate of the error term $B(t,x)(g(t,x,\theta) - g(t,x,\hat{\theta}))$. To ensure this, one may introduce control in the form of nonlinear damping with a sufficiently high growth rate, similar to the technique used in adaptive backstepping (Krstić et al., 1995). We also refer to Panteley and Loría (2001) for an extensive discussion on how to ensure UGB. Note that controllability of the system depends on the properties of $f(t,x)$ and $B(t,x)$.

Theorem 2 *Suppose that Assumptions 1–3 hold such that $a_3(x) \geq a_3^* > 0$. Then for each compact neighborhood $K' \subset \mathbb{R}^{2n}$ of the origin, there exists $k_\phi > 0$ such that, if K_ϕ is chosen such that $\lambda_{\min}(K_\phi) > k_\phi$, then the origin of (15), (11) is uniformly asymptotically stable with $K' \times (\Theta - \Theta)$ contained in the region of attraction.* □

PROOF This proof is based on the proof of Panteley and Loría (2001, Lemma 2). The UGAS property of the unperturbed system, together with the fact that $f(t,x)$ is locally Lipschitz continuous in x, uniformly in t, and continuously differentiable on $\mathbb{R}_{\geq 0} \times \mathbb{R}^n$, implies by Panteley and Loría (2001, Prop. 1) the existence of a Lyapunov function $V_x(t,x)$; class $\mathcal{K}_\infty$ functions α_1 and α_2; and a class $\mathcal{K}$ function α_4 such that, for all $(t,x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$,

$$\alpha_1(\|x\|) \leq V_x(t,x) \leq \alpha_2(\|x\|), \tag{16}$$

$$\frac{\partial V_x}{\partial t}(t,x) + \frac{\partial V_x}{\partial x}(t,x) f(t,x) \leq -V_x(t,x), \tag{17}$$

$$\left\| \frac{\partial V_x}{\partial x}(t,x) \right\| \leq \alpha_4(\|x\|). \tag{18}$$

As before, we know that $\hat{\theta}(t_0) \in \Theta$ implies that $\hat{\theta}(t)$ remains in Θ. Let $R > 0$ be chosen large enough that $\Omega := \{(x,\xi) \mid \|(x,\xi)\| \leq R\} \supset K' \times (\Theta - \Theta)$. For any $0 < r \leq R$, we know that if $\|(x(t_0), \xi(t_0))\| \leq r$, then $\|x(t_0)\| \leq r$, which by the UGB property from Assumption 3 implies that there exists a $c_x(r) > 0$ such that, for all $t \geq t_0$, $\|x(t)\| \leq c_x(r)$. It follows that, for all $(x(t_0), \xi(t_0)) \in \Omega$ such that $\xi(t_0) \in \Xi$, $x(t)$ is restricted to some compact set K. Let therefore $\lambda_{\min}(K_\phi)$ be chosen large enough to ensure exponential stability of the estimator according to Theorem 1 based on the set K. Then we know that $\|\xi(t)\| \leq k_e\|\xi(t_0)\|\exp(-\lambda(t-t_0))$. This implies that, if $\|(x(t_0), \xi(t_0))\| \leq r$ and $\xi(t_0) \in \Xi$, then $\|(x(t), \xi(t))\| \leq c(r)$, where $c(r) := (c_x^2(r) + (k_e r)^2)^{1/2}$.

Define $v_x(t) = V_x(t, x(t))$. We then have $\dot{v}_x(t) \leq -v_x(t) + \alpha_4(c(r))\beta(r, t-t_0)$, where $\beta(r, t-t_0) := \gamma(k_e r \exp(-\lambda(t-t_0)))$ is a class $\mathcal{KL}$ function by Khalil (2001, Lemma 4.2). Let $\tau_0 \geq t_0$. Multiplying by $\exp(t-\tau_0)$ on both sides and rearranging, we have, for all $t \geq \tau_0$, $\frac{\mathrm{d}}{\mathrm{d}t}(v_x(t)\exp(t-\tau_0)) \leq \alpha_4(c(r))\beta(r, t-t_0)\exp(t-\tau_0)$. Integrating from τ_0 to t on both sides and multiplying by $\exp(-(t-\tau_0))$, we have $v_x(t) \leq v_x(\tau_0)\exp(-(t-\tau_0)) + \alpha_4(c(r))\int_{\tau_0}^{t}\exp(-(t-s))\beta(r, s-t_0)\,\mathrm{d}s$, which means that replacing τ_0 with t_0 in the above expression yields, for all $t \geq t_0$, $v_x(t) \leq v_x(t_0)\exp(-(t-t_0)) + \alpha_4(c(r))\beta(r,0)\int_{t_0}^{t}\exp(-(t-s))\,\mathrm{d}s \leq v_x(t_0) + \alpha_4(c(r))\beta(r,0)(1 - \exp(-(t-t_0))) \leq \gamma'(r)$, where $\gamma'(r) := \alpha_2(r) + \alpha_4(c(r))\beta(r,0)$. Hence, $\|x(t)\| \leq \alpha_1^{-1}(\gamma'(r))$, and $\alpha_1^{-1} \circ \gamma'$ is a class $\mathcal{K}_\infty$ function by Khalil (2001, Lemma 4.2). Furthermore, we have, for $\|(x(t_0), \xi(t_0))\| \leq r$ and $\xi(t_0) \in \Xi$, $\|(x(t), \xi(t))\| \leq \gamma''(r)$, where $\gamma''(r) := ((\alpha_1^{-1}(\gamma'(r)))^2 + (k_e r)^2)^{1/2}$ is a class $\mathcal{K}_\infty$ function. Let $c \leq R$ be sufficiently small such that $\|\xi\| \leq c \implies \xi \in \Xi$. By the above, we have that, for all $\|(x(t_0), \xi(t_0))\| \leq r < c$ and for all $t \geq t_0$, $\|(x(t), \xi(t))\| \leq \gamma''(r)$, which means that the origin of (15), (11) is uniformly stable.

For some $\varepsilon_1 > 0$, define T_1 large enough that $\alpha_4(c(r))\beta(r, T_1) \leq \frac{1}{2}\varepsilon_1$. Substituting $\tau_0 = t_0 + T_1$ into the earlier bound on

$v_x(t)$, we obtain that, $\forall t \geq t_0 + T_1$,

$$\begin{aligned} v_x(t) &\leq v_x(t_0+T_1)\mathrm{e}^{-(t-t_0-T_1)} \\ &\quad + \alpha_4(c(r))\int_{t_0+T_1}^{t} \beta(r, s-t_0)\mathrm{e}^{-(t-s)}\,\mathrm{d}s \\ &\leq \gamma'(r)\mathrm{e}^{-(t-t_0-T_1)} + \frac{\varepsilon_1}{2}. \end{aligned} \tag{19}$$

Now let $T_2 \geq T_1$ be chosen large enough that $\gamma'(r)\exp(-(T_2 - T_1)) \leq \frac{1}{2}\varepsilon_1$. Then we have, for all $t \geq t_0 + T_2$, $v_x(t) \leq \gamma'(r)\exp(-(T_2-T_1)) + \frac{1}{2}\varepsilon_1 \leq \varepsilon_1$. Hence, for all $t \geq t_0 + T_2$, $\|x(t)\| \leq \alpha_1^{-1}(\varepsilon_1)$. Define ε such that $\varepsilon_1 = \alpha_1(\varepsilon/\sqrt{2})$ and let $T \geq T_2$ be large enough that $k_\mathrm{e} r\exp(-\lambda T) \leq \varepsilon/\sqrt{2}$. Then $\forall t \geq t_0 + T$, $\|(x(t), \xi(t))\| \leq (\frac{1}{2}\varepsilon^2 + \frac{1}{2}\varepsilon^2)^{1/2} = \varepsilon$. Since ε can be chosen arbitrarily small, and the above holds for all initial conditions such that $(x(t_0), \xi(t_0)) \in \Omega$ and $\xi(t_0) \in \Xi$, it follows that the whole system (15), (11) is uniformly asymptotically stable with $K' \times (\Theta - \Theta)$ contained in the region of attraction. ■

Remark 4 Theorems 1 and 2 are intended to show that particular stability properties are guaranteed by choosing the gain K_ϕ sufficiently high; they are not intended as a practical guide to tuning the estimator gains. Attempting to find a numerical value for k_ϕ, the lower bound on the eigenvalues of K_ϕ, is likely to be complicated and of little practical use, owing to the conservative nature of Lyapunov-type analysis. In practical implementations, the gains are normally found through a tuning procedure involving simulations or tests with the actual system. □

4.1 Vanishing Excitation at $x = 0$

So far we have only considered perturbations that are persistently excited in the sense that θ can always be estimated from ϕ with exponential convergence rate. This strict requirement excludes a class of perturbations where we have persistent excitations as long as the controlled state x is bounded away from the origin, but where the excitation is lost at the origin. Most importantly, this includes all perturbations that vanish for $x = 0$. As an example, consider the system $\dot{x} = -x + \arctan(\theta x) - \arctan(\hat{\theta}x)$. In the following theorem, we show that, under certain conditions, convergence of the controlled state to the origin is guaranteed even when excitation is lost at the origin.

Theorem 3 *Suppose that Assumptions 1–3 hold such that $(L_1(x) + L_2(x))^2 \leq \rho a_3(x)$ for some number $\rho > 0$, locally around the origin. Then, for each compact neighborhood $K' \subset \mathbb{R}^{2n}$ of the origin, there exists $k_\phi > 0$ such that, if K_ϕ is chosen such that $\lambda_{\min}(K_\phi) > k_\phi$ and the trajectory of* (15), (11) *originates in $K' \times (\Theta - \Theta)$, then* $\lim_{t\to\infty} x(t) = 0$ *and $\xi(t)$ is bounded.* □

PROOF As in the proof of Theorem 2, the UGB condition in Assumption 3 ensures that, for trajectories originating in $K' \times (\Theta - \Theta)$, the state $x(t)$ remains in a compact set K. To investigate what happens to the estimator, we follow the proof of Theorem 1, where we find that we have the requirement $a_3(x)\lambda_{\min}(K_\phi) > \frac{1}{4}(a_4 L_2(x) + L_1(x))^2$. Because $L_1(x)$ and $L_2(x)$ are bounded on any compact set, and due to the local condition around $x = 0$ in Theorem 3, the inequality can be satisfied outside the origin for $\lambda_{\min}(K_\phi) > k_\phi$, for some $k_\phi > 0$. This results in $\dot{V}_\mathrm{p}(t,\xi) \leq -\zeta^\top Q(x)\zeta$, where $\zeta = [\|\tilde{\phi}\|, \|\tilde{\theta}\|]^\top$, and where $Q(x)$ is positive definite for each $x \neq 0$, and positive semidefinite for $x = 0$. Define $U(x) = \lambda_{\min}(Q(x))/\max\{2a_2, 1\}$, which is a continuous positive-definite function (due to continuity of the eigenvalues and of $a_3(x)$, $L_1(x)$ and $L_2(x)$). Following the same argument as in the proof of Theorem 1, we can then write $\|\xi(t)\| \leq \beta(t) := k_\mathrm{e}\|\xi(t_0)\|\exp(-\int_{t_0}^{t} U(x(\tau))\,\mathrm{d}\tau)$. Hence, β is a monotonically non-increasing function.

For the sake of establishing a contradiction, suppose that $x(t)$ does not converge to the origin. Then there exists a $\delta > 0$ such that, for all $t \geq t_0$, there exist $\tau \geq t$ such that $\|x(\tau)\| \geq 2\delta$. From Assumption 3, $\|B(t,x)(g(t,x,\theta) - g(t,x,\hat{\theta}))\|$ is uniformly bounded when $\|x(t)\| \in [\delta, 2\delta]$, and the same holds for $\|f(t,x)\|$, because $f(t,x)$ is locally Lipschitz continuous in x, uniformly in t, and $f(t,0) = 0$. Hence, the right-hand side of (15) is uniformly bounded for $\|x(t)\| \in [\delta, 2\delta]$, and it follows that there exists $T > 0$ such that on each interval $[\tau - T, \tau + T]$, $\|x(t)\| \geq \delta$. On every such interval there is a decrease in the bounding function β; in particular, $\beta(\tau + T) \leq \beta(\tau - T)\exp(-2\bar{\lambda}T)$, where $\bar{\lambda} = \min_{x \in K \setminus B(\delta)} U(x)$ is a positive number. Moreover, for any integer $n > 0$, there exists a $t_1 > t_0$ such that $[t_0, t_1]$ contains at least n disjoint time intervals of length $2T$ with $\|x(t)\| \geq \delta$. The UGAS and UGB properties of the unperturbed system $\dot{x} = f(t,x)$ imply that if $\gamma(\|\tilde{\theta}(t)\|)$ is sufficiently small for all $t \geq t_1$, where $t_1 \geq t_0$ is arbitrary, then $\|x(t)\|$ is globally ultimately bounded by δ. Let therefore ε be chosen small enough that, if for all $t \geq t_1$, $\|\xi(t)\| \leq \varepsilon$, then $\|x(t)\|$ is globally ultimately bounded by δ. Let $n \geq 0$ be an integer chosen large enough that $\beta(t_0)\exp(-2n\bar{\lambda}T) \leq \varepsilon$, and let t_1 be large enough that there are at least n disjoint intervals of length $2T$ in $[t_0, t_1]$ with $\|x(t)\| \geq \delta$. This implies that, for all $t \geq t_1$, $\|\xi(t)\| \leq \varepsilon$. This, in turn, implies by the ultimate boundedness property that there exists a $t_2 \geq t_1$ such that, for all $t \geq t_2$, $\|x(t)\| \leq \delta$. But this contradicts our assumption that there exist arbitrarily large values τ such that $\|x(\tau)\| \geq 2\delta$. Hence, $x(t)$ does converge to the origin. ■

The functions $L_1(x)$ and $L_2(x)$ represent Lipschitz-like bounds that are typically not derived explicitly in the design process. The condition in Theorem 3 concerns the growth rates of these functions as $x \to 0$, which can often be determined without developing explicit expressions for the functions.

5 Simulation Example

In the next example, we demonstrate the method on a first-order system with a nonlinear and time-varying perturbation.

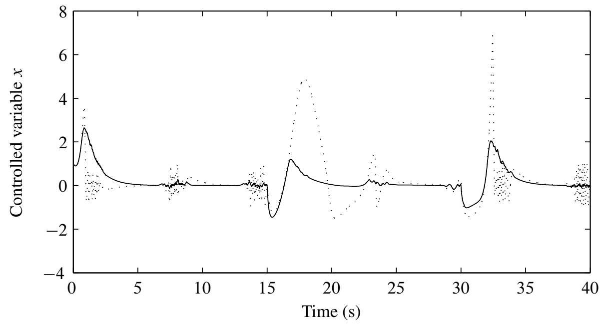
Example 6 Consider the system

$$\dot{x} = -x + \mathrm{e}^{\sin(t)\theta} + u, \tag{20}$$

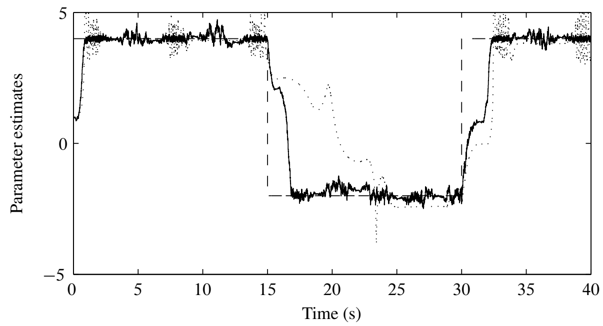
where $\theta \in [\theta_{\min}, \theta_{\max}] = [-5, 5]$. Here $f(t,x) = f(x) = -x$, $B(t,x) = 1$, and $g(t,x,\theta) = g(t,\theta) = \exp(\sin(t)\theta)$. We wish to use u to cancel the perturbation, and we therefore let $u = -\exp(\sin(t)\hat{\theta})$. The first step is to design an update law to estimate θ from the full perturbation. We first note that $[\partial g/\partial\theta](t,\theta) = \sin(t)\exp(\sin(t)\theta)$, and hence (7) in Proposition 4 is satisfied by selecting $M(t,x,\hat{\theta}) = M(t) = \sin(t)$ with $S(t,x) = S(t) = \sin^2(t)\exp(-\theta')$, where $\theta' := \max_{\theta\in\Theta}|\theta|$. The remaining requirements in Proposition 4 can be confirmed in the same way as in Example 4. We now check that the conditions of Assumption 2 hold. We have that $d(t,x,\tilde{\theta}) = (\theta\exp(\sin(t)\theta) - \hat{\theta}\exp(\sin(t)\hat{\theta}))\cos(t)$. Using the mean value theorem, we find that $|d(t,x,\tilde{\theta})| \leq (1+\theta')\exp(\theta')|\tilde{\theta}|$. We also see that $|u_\theta(t,x,\phi,\hat{\theta}) - u_\theta(t,x,\hat{\phi},\hat{\theta})| = \Gamma|\sin(t)\tilde{\phi}| \leq \Gamma|\tilde{\phi}|$.[1] Moving to Assumption 3, it is straightforward to see that the nominal, unperturbed system $\dot{x} = -x$ is UGAS and that the perturbed system is UGB when $\hat{\theta}$ is restricted to Θ. Finally, we use $\gamma(s) = \exp(\theta')s$ to satisfy Assumption 3. We implement the full estimator from (8). After canceling terms, we obtain

$$\begin{aligned}\dot{z} &= -K_\phi(K_\phi - 1)x - K_\phi z \\ &\quad - \sin(t)\mathrm{e}^{\sin(t)\hat{\theta}}\mathrm{Proj}(\Gamma\sin(t)(z + K_\phi x)), \qquad (21)\\ \dot{\hat{\theta}} &= \mathrm{Proj}(\Gamma\sin(t)(z + K_\phi x)). \qquad (22)\end{aligned}$$

We simulate the system with $K_\phi = 10$ and $\Gamma = 3$, letting θ vary in steps between -2 and 4 to get an impression of the response. The results can be seen in Figure 1, where we have also plotted the response using a gradient-type algorithm $\dot{\hat{\theta}} = \Gamma\sin(t)\exp(\sin(t)\hat{\theta})x$ based on linearization around the estimate $\hat{\theta}$, with gain $\Gamma = 1$ and with the parameter estimate limited to $[-5,5]$. Noise has been added with sample time 0.001 and variance 1 to the measurement of the state x used in both algorithms. The parameter projection is not active at any point during the simulation. □



(a) Controlled variable, nonlinear method (solid), and gradient-type method (dotted)



(b) Unknown parameter (dashed), estimate with nonlinear method (solid), and estimate with gradient-type method (dotted)

Fig. 1. Simulation results for Example 6

6 Application: Downhole Pressure Estimation During Oil Well Drilling

When extracting hydrocarbons from underground geological formations it is usually necessary to create a well by drilling a wellbore. During drilling, a mud circulation system is used to transport cuttings from the drilling out of the wellbore. The mud is pumped downhole inside the drill string and through the drill bit, and returns to the top through the annulus containing the drill string. The downhole pressure needs to be controlled within its margins: above the reservoir pore pressure and wellbore collapse pressure, but below the wellbore fracture pressure. In many cases, this margin is quite wide and the pressure can be manually controlled, but as oil and gas reserves begin to be depleted, reservoirs with narrower margins are being drilled, demanding automated pressure control (see, e.g., Nygaard and Nævdal, 2006; Nygaard, Imsland, and Johannessen, 2007). The downhole pressure is usually measured, but with conventional equipment this measurement has low bandwidth and is unreliable. Good pressure control therefore demands pressure estimation based on topside measurements.

6.1 Modeling

Complex models of the drilling process exist, for example, in the simulator Wemod, provided by IRIS (Lage, Frøyen, Sævareid, and Fjelde, 2000). Here we shall use a low-complexity model for the development of the pressure estimation algorithm (see Stamnes, Zhou, Kaasa, and Aamo, 2008). In particular, we assume that the drilling process is described by the following dynamic model, derived from mass balances for the drill string and annulus:

$$\frac{V_d}{\beta_d}\dot{p}_p = q_p - q_b, \tag{23}$$

$$\frac{V_a}{\beta_a}\dot{p}_c = -\dot{V}_a + q_b + q_r + q_a - q_c, \tag{24}$$

[1] We recall from Remark 2 that we can disregard the projection when checking this condition.

where the states p_p and p_c are the pressures in the top of the drill string (standpipe pressure) and the annulus (choke pressure), both of which are measured. Furthermore, V_d and V_a denote the volumes of the drill string and the annulus; and β_d and β_a are the drill string and annulus bulk moduli, all known. The volume flows are the inflow to the drill string (q_p), flow from the back pressure (annulus) pump (q_a), and exit flow from the annulus through the choke (q_c), all measured, as well as the flow through the drill bit (q_b) and inflow from the reservoir (q_r). The flow q_b is given by a steady-state momentum balance for the drill string and annulus (in a slight simplification of the model in Stamnes et al. (2008)):

$$p_p - p_c = F_d q_b^2 + F_a(q_b + q_r)^2 - s(t). \tag{25}$$

The friction parameter F_d in the drill string is assumed known, as is the function $s(t) = (\rho_d(t) - \rho_a(t))gh_b(t)$, which describes the difference in drill string and annulus downhole static head. We shall estimate the two remaining parameters, the friction coefficient F_a and the reservoir inflow q_r, which will allow us to calculate the downhole pressure p_b using a steady-state momentum balance for the annulus as $p_b = p_c + F_a(q_b + q_r)^2 + \rho_a(t)gh_b$. We assume that the parameters to be estimated are constant, and that $(q_b + q_r)^2 > \alpha$ for some $\alpha > 0$, which implies that we have flow into the annulus. In order to put the system in the form used in this paper, we write $x = [V_d/\beta_d p_p, V_a/\beta_a p_c]^\top$, $\theta = [q_r, F_a]^\top$, $f(t,x) = [q_p, (x_2/V_a - 1)\dot{V}_a + q_a - q_c]^\top$, $B(t,x) = \left[\begin{smallmatrix} -1 & 0 \\ 1 & 1 \end{smallmatrix}\right]$, and $g(t,x,\theta) = [q_b, q_r]^\top$.

6.2 Estimator Design

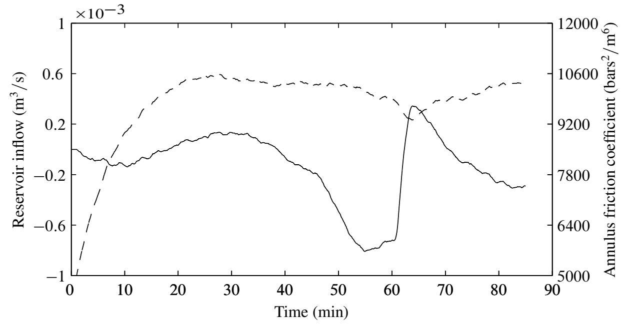
As before, we start by designing an update law for estimating F_a and q_r as if $\phi_1 = -q_b$ and $\phi_2 = q_b + q_r$ were known. We see that we can use a simple inversion according to Proposition 1 to create an update law for q_r:

$$\dot{\hat{q}}_r = \Gamma_1(\hat{\phi}_1 + \hat{\phi}_2 - \hat{q}_r), \tag{26}$$

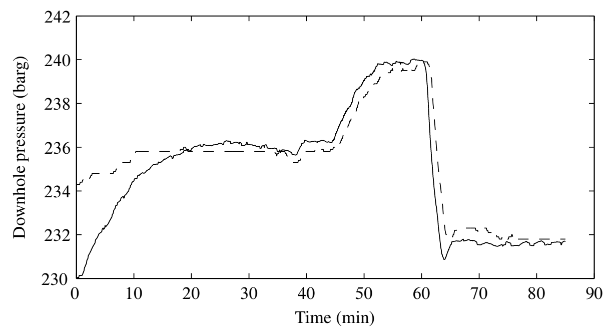
where $\Gamma_1 > 0$ is a scalar gain. (For simplicity, we omit the projection in discussing this example.) For F_a, the approach is slightly more complicated. According to (25), we may define an estimated flow $\hat{q}_b$ through the bit, by the equation $p_p - p_c = F_d \hat{q}_b^2 + \hat{F}_a(\hat{q}_b + \hat{q}_r)^2 - s(t)$. Subtracting this from (25) and rearranging yields the relation $-F_d(q_b^2 - \hat{q}_b^2) - \hat{F}_a\left((q_b + q_r)^2 - (\hat{q}_b + \hat{q}_r)^2\right) = \tilde{F}_a(q_b + q_r)^2$. Define the update law

$$\dot{\hat{F}}_a = \Gamma_2[-F_d(\hat{\phi}_1^2 - \hat{q}_b^2) - \hat{F}_a(\hat{\phi}_2^2 - (\hat{q}_b + \hat{q}_r)^2)]. \tag{27}$$

For $\hat{\phi} = \phi$, we then have $\dot{\tilde{F}}_a = -\Gamma_2 \tilde{F}_a(q_b + q_r)^2$. It is then straightforward to prove that Assumption 1 holds with $V_u(\tilde{\theta}) = \frac{1}{2}\tilde{\theta}^\top \Gamma^{-1}\tilde{\theta}$, where Γ is the gain matrix composed of Γ_1 and Γ_2. Implementation of the update law requires calculation of $\hat{q}_b$. We find $\hat{q}_b$ by taking the positive root of the second-order equation defining the estimated flow through the bit, which we assume is always real. This solution is



(a) Estimated reservoir inflow q_r (solid, left axis) and estimated friction coefficient F_a (dashed, right axis)



(b) Measured (dashed) and estimated (solid) downhole pressure p_b

Fig. 2. Results for drilling application using real drilling data

in turn used to find the partial derivative $[\partial g/\partial\theta](t,x,\hat{\theta})$, which is needed in the complete implementation of the system. Due to the quadratic terms in ϕ_1 and ϕ_2 in the update law for F_a, the Lipschitz condition on u_θ does not hold globally. This problem can easily be fixed by modifying the update law with a saturation, as described in Section 3.3.

6.3 Experimental Results

The estimator has been tested using the simulator Wemod (Lage et al., 2000), yielding very accurate results, and on real measurement data from drilling at the Grane field in the North Sea. The results for the real drilling data can be seen in Figure 2. The tuning used is $\Gamma_1 = 0.005$, $\Gamma_2 = 2$ and $K_\phi = 10I$. It should be noted that, although it is common to measure the flow q_c, no such measurement is available in the data set used, and q_c is therefore estimated from a choke model and the available choke opening. Given the large uncertainties in this application, the downhole pressure estimate is considered good.

7 Concluding Remarks

We have introduced a modular design method for estimating unknown parameters in systems with nonlinearly parameterized perturbations, where the main design task is to construct an update law to asymptotically invert a nonlinear equation. As mentioned in the introduction, the modular structure allows for the perturbation estimator to be extended beyond what is presented in this paper. In Grip et al. (2009), it is extended to facilitate observer design for the case of partial state measurement, by using techniques from high-gain observer theory. We note that, for the results presented in this paper, we can write (8) in terms of a variable $\hat{x} = -K_\phi^{-1} z$ rather than z. It is then easy to see that $\hat{x}$ represents an estimate of the state x. It can furthermore be confirmed that, for linearly parameterized perturbations, the design closely resembles a standard linear observer with adaptation, if the update law is chosen as suggested in Remark 1.

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A Parameter Projection

Let the set of possible parameters be defined by $\Pi := \{\hat{\theta} \in \mathbb{R}^p \mid \mathscr{P}(\hat{\theta}) \le 0\}$, where $\mathscr{P} \colon \mathbb{R}^p \to \mathbb{R}$ is a smooth, convex function. Let Π^0 denote the interior of Π, and let Θ be defined by $\Theta = \{\hat{\theta} \in \mathbb{R}^p \mid \mathscr{P}(\hat{\theta}) \le \varepsilon\}$, where ε is a small positive number, making Θ a slightly larger superset of Π. Consider the update function $u_\theta(t,x,\hat{\phi},\hat{\theta}) = \mathrm{Proj}(\tau(t,x,\hat{\phi},\hat{\theta}))$, where $\mathrm{Proj}(\cdot)$ is the projection from Krstić et al. (1995, Appendix E). Proj is defined as $\mathrm{Proj}(\tau(t,x,\hat{\phi},\hat{\theta})) = p(t,x,\hat{\phi},\hat{\theta})\tau(t,x,\hat{\phi},\hat{\theta})$, with $p(t,x,\hat{\phi},\hat{\theta})$ given by

- $p(t,x,\hat{\phi},\hat{\theta}) = I$ if $\hat{\theta} \in \Pi^0$ or $\nabla_{\hat{\theta}}\mathscr{P}^\top \tau(t,x,\hat{\phi},\hat{\theta}) \le 0$,
- $p(t,x,\hat{\phi},\hat{\theta}) = \left(I - c(\hat{\theta})\Gamma\nabla_{\hat{\theta}}\mathscr{P}\nabla_{\hat{\theta}}\mathscr{P}^\top / \|\nabla_{\hat{\theta}}\mathscr{P}\|_\Gamma^2\right)$ if $\hat{\theta} \in \Theta \setminus \Pi^0$ and $\nabla_{\hat{\theta}}\mathscr{P}^\top \tau(t,x,\hat{\phi},\hat{\theta}) > 0$,

where Γ is a symmetric positive-definite matrix corresponding to the gain matrix in the update law; $\nabla_{\hat{\theta}}\mathscr{P}^\top$ is the gradient of $\mathscr{P}(\hat{\theta})$ with respect to $\hat{\theta}$; and $c(\hat{\theta}) = \min\{1, \mathscr{P}(\hat{\theta})/\varepsilon\}$.

A.1 Lipschitz Continuity

We wish to show that, if, for each compact set $K \in \mathbb{R}^n$, τ has the property that, for all $(t,x,\phi,\hat{\phi},\hat{\theta}) \in \mathbb{R}_{\ge 0} \times K \times \mathbb{R}^n \times \mathbb{R}^n \times \Theta$, $\|\tau(t,x,\phi,\hat{\theta}) - \tau(t,x,\hat{\phi},\hat{\theta})\| \le L_2(x)\|\tilde{\phi}\|$, then we also have $\|u_\theta(t,x,\phi,\hat{\theta}) - u_\theta(t,x,\hat{\phi},\hat{\theta})\| \le L_2'(x)\|\tilde{\phi}\|$, for some continuous function $L_2'(x) > 0$. In the following, we shall outline the proof of this assertion. To do this, we have to look at two distinct cases: when the parameter projection is either active or inactive for both $u_\theta(t,x,\phi,\hat{\theta})$ and $u_\theta(t,x,\hat{\phi},\hat{\theta})$ (Case I); and when the parameter projection is active for one of $u_\theta(t,x,\phi,\hat{\theta})$ or $u_\theta(t,x,\hat{\phi},\hat{\theta})$, but not the other (Case II). In the following, we shall write $u_\theta(\phi) = u_\theta(t,x,\phi,\hat{\theta})$, $u_\theta(\hat{\phi}) = u_\theta(t,x,\hat{\phi},\hat{\theta})$, and similarly for τ.

In Case I, $p(t,x,\phi,\hat{\theta}) = p(t,x,\hat{\phi},\hat{\theta})$. The property therefore follows from uniform boundedness of $\|p(t,x,\phi,\hat{\theta})\|$. Case II occurs if $\hat{\theta} \in \Theta \setminus \Pi^0$, and $\nabla_{\hat{\theta}}\mathscr{P}^\top\tau(\phi)$ and $\nabla_{\hat{\theta}}\mathscr{P}^\top\tau(\hat{\phi})$ do not have the same sign. Without loss of generality, we assume that $\nabla_{\hat{\theta}}\mathscr{P}^\top\tau(\phi) \le 0$ and $\nabla_{\hat{\theta}}\mathscr{P}^\top\tau(\hat{\phi}) > 0$. In this case, we have $u_\theta(\phi) - u_\theta(\hat{\phi}) = \tau(\phi) - (I - c(\hat{\theta})\Gamma\nabla_{\hat{\theta}}\mathscr{P}\nabla_{\hat{\theta}}\mathscr{P}^\top/\|\nabla_{\hat{\theta}}\mathscr{P}\|_\Gamma^2)\tau(\hat{\phi})$. Expanding this expression, we have, after some calculation, $\|u_\theta(\phi) - u_\theta(\hat{\phi})\|_{\Gamma^{-1}}^2 = \|\tau(\phi) - \tau(\hat{\phi})\|_{\Gamma^{-1}}^2 + c(\hat{\theta})/\|\nabla_{\hat{\theta}}\mathscr{P}\|_\Gamma^2 \big[c(\hat{\theta})|\nabla_{\hat{\theta}}\mathscr{P}^\top\tau(\hat{\phi})|^2 + 2\tau(\hat{\phi})^\top\nabla_{\hat{\theta}}\mathscr{P}\nabla_{\hat{\theta}}\mathscr{P}^\top(\tau(\phi) - \tau(\hat{\phi}))\big]$. We now make the observation that, because $\nabla_{\hat{\theta}}\mathscr{P}^\top\tau(\phi)$ and $\nabla_{\hat{\theta}}\mathscr{P}^\top\tau(\hat{\phi})$ do not have the same sign, $|\nabla_{\hat{\theta}}\mathscr{P}^\top\tau(\hat{\phi})| \le |\nabla_{\hat{\theta}}\mathscr{P}^\top(\tau(\phi) - \tau(\hat{\phi}))|$. Using this for substitution where $\nabla_{\hat{\theta}}\mathscr{P}^\top\tau(\hat{\phi})$ occurs alone, we obtain that $\|u_\theta(\phi) - u_\theta(\hat{\phi})\|_{\Gamma^{-1}}^2 \le \|\tau(\phi) - \tau(\hat{\phi})\|_{\Gamma^{-1}}^2 + (c^2(\hat{\theta}) + 2c(\hat{\theta}))/\|\nabla_{\hat{\theta}}\mathscr{P}\|_\Gamma^2\|\nabla_{\hat{\theta}}\mathscr{P}\|^2\|\tau(\phi) - \tau(\hat{\phi})\|^2$. Using the property that for $P = P^\top > 0$, $\lambda_{\min}(P)\|\zeta\|^2 \le \|\zeta\|_P^2 \le \lambda_{\max}(P)\|\zeta\|^2$, we find that $\|u_\theta(\phi) - u_\theta(\hat{\phi})\| \le \alpha\|\tau(\phi) - \tau(\hat{\phi})\| \le \alpha L_2(x)\|\tilde{\phi}\|$, where $\alpha = [(\lambda_{\max}(\Gamma^{-1})\lambda_{\min}(\Gamma) + 3)/(\lambda_{\min}(\Gamma^{-1})\lambda_{\min}(\Gamma))]^{1/2}$.

B Proofs of Propositions 2–4

PROOF (PROPOSITION 2) Inspired by Loría, Panteley, Popović, and Teel (2005), we use the function $V_{\mathrm{u}}(t,\tilde{\theta}) = \frac{1}{2}\tilde{\theta}^\top\left(\Gamma^{-1} - \mu\int_t^\infty \exp(t-\tau)Il(\tau,x(\tau))\,\mathrm{d}\tau\right)\tilde{\theta}$, where $\mu > 0$ is a constant yet to be specified. We first note that $\frac{1}{2}\tilde{\theta}^\top\left(\Gamma^{-1} - \mu I\right)\tilde{\theta} \le V_{\mathrm{u}}(t,\tilde{\theta}) \le \frac{1}{2}\tilde{\theta}^\top\Gamma^{-1}\tilde{\theta}$. Hence, V_{u} is positive definite provided that $\mu < \lambda_{\min}(\Gamma^{-1})$. With $\hat{\phi} = \phi$, we get $\dot{\tilde{\theta}} = -\mathrm{Proj}(l(t,x)\Gamma\tilde{\theta})$. Using the property (Krstić et al., 1995, Lemma E.1) that $-\tilde{\theta}^\top\Gamma^{-1}\mathrm{Proj}(\tau) \le -\tilde{\theta}^\top\Gamma^{-1}\tau$, we have

$$
\begin{aligned}
\dot{V}_{\mathrm{u}}(t,\tilde{\theta}) &= -\tilde{\theta}^\top\left(\Gamma^{-1} - \mu\int_t^\infty \mathrm{e}^{t-\tau}Il(\tau,x(\tau))\,\mathrm{d}\tau\right) \\
&\quad \cdot \mathrm{Proj}(l(t,x)\Gamma\tilde{\theta}) + \frac{1}{2}\mu\tilde{\theta}^\top Il(t,x)\tilde{\theta} \\
&\quad - \frac{1}{2}\mu\tilde{\theta}^\top\int_t^\infty \mathrm{e}^{t-\tau}Il(\tau,x(\tau))\,\mathrm{d}\tau\,\tilde{\theta} \\
&\le -(1 - \frac{1}{2}\mu)l(t,x)\tilde{\theta}^\top\tilde{\theta} - \frac{1}{2}\mu\varepsilon\mathrm{e}^{-T}\tilde{\theta}^\top\tilde{\theta} \\
&\quad + \mu\|\tilde{\theta}\|\left\|\int_t^\infty \mathrm{e}^{t-\tau}Il(\tau,x(\tau))\,\mathrm{d}\tau\right\|\|\mathrm{Proj}(l(t,x)\Gamma\tilde{\theta})\| \\
&\le -(1 - \frac{1}{2}\mu - \mu\sqrt{\kappa}\|\Gamma\|)l(t,x)\|\tilde{\theta}\|^2 - \frac{1}{2}\mu\varepsilon\mathrm{e}^{-T}\|\tilde{\theta}\|^2,
\end{aligned}
\tag{B.1}
$$

where κ is the ratio of the largest to the smallest eigenvalue of Γ^{-1}. Above, we have used the property (Krstić et al., 1995, Lemma E.1) that $\mathrm{Proj}(\tau)^\top\Gamma^{-1}\mathrm{Proj}(\tau) \le \tau^\top\Gamma^{-1}\tau$, which implies that $\|\mathrm{Proj}(\tau)\| \le \sqrt{\kappa}\|\tau\|$. We have also used that $\int_t^\infty \exp(t-\tau)l(\tau,x(\tau))\,\mathrm{d}\tau \ge \int_t^{t+T}\exp(t-\tau)l(\tau,x(\tau))\,\mathrm{d}\tau \ge \exp(-T)\int_t^{t+T} l(\tau,x(\tau))\,\mathrm{d}\tau \ge \exp(-T)\varepsilon$. From the calculation above, we see that the time derivative is negative definite provided that $\mu < 1/(\frac{1}{2} + \sqrt{\kappa}\|\Gamma\|)$. ■

PROOF (PROPOSITION 3) For the sake of brevity, we write $M = M(t,x,\hat{\theta})$ and $B = B(t,x)$. With $\hat{\phi} = \phi$, we get $\dot{\tilde{\theta}} = -\text{Proj}(\Gamma MB(g(t,x,\theta) - g(t,x,\hat{\theta})))$. We use the function $V_{\text{u}}(t,\tilde{\theta}) = \frac{1}{2}\tilde{\theta}^{\mathsf{T}}\Gamma^{-1}\tilde{\theta}$. Using the property (Krstić et al., 1995, Lemma E.1) that $-\tilde{\theta}^{\mathsf{T}}\Gamma^{-1}\text{Proj}(\tau) \leq -\tilde{\theta}^{\mathsf{T}}\Gamma^{-1}\tau$, we have $\dot{V}_{\text{u}}(t,\tilde{\theta}) \leq -\frac{1}{2}\tilde{\theta}^{\mathsf{T}}MB(g(t,x,\theta) - g(t,x,\hat{\theta})) - \frac{1}{2}(g(t,x,\theta) - g(t,x,\hat{\theta}))^{\mathsf{T}}B^{\mathsf{T}}M^{\mathsf{T}}\tilde{\theta}$. Since $g(t,x,\theta)$ is continuously differentiable with respect to θ, we may write, according to Taylor's theorem (see, e.g., Nocedal and Wright, 1999, Theorem 11.1), $g(t,x,\theta) - g(t,x,\hat{\theta}) = \int_0^1 [\partial g/\partial\theta](t,x,\hat{\theta} + p\tilde{\theta})\tilde{\theta}\,\mathrm{d}p$. Hence, we have $\dot{V}_{\text{u}}(t,\tilde{\theta}) \leq -\frac{1}{2}\int_0^1 \tilde{\theta}^{\mathsf{T}}(MB[\partial g/\partial\theta](t,x,\hat{\theta} + p\tilde{\theta}) + [\partial g/\partial\theta]^{\mathsf{T}}(t,x,\hat{\theta} + p\tilde{\theta})B^{\mathsf{T}}M^{\mathsf{T}})\tilde{\theta}\,\mathrm{d}p \leq -\int_0^1 \tilde{\theta}^{\mathsf{T}}P\tilde{\theta}\,\mathrm{d}p = -\tilde{\theta}^{\mathsf{T}}P\tilde{\theta}$, which proves that Assumption 1 holds. ■

PROOF (PROPOSITION 4) We use the function $V_{\text{u}}(t,\tilde{\theta}) = \frac{1}{2}\tilde{\theta}^{\mathsf{T}}\left(\Gamma^{-1} - \mu\int_t^\infty \exp(t-\tau)S(\tau,x(\tau))\,\mathrm{d}\tau\right)\tilde{\theta}$, where $\mu > 0$ is a constant yet to be specified. First, we confirm that the function V_{u} is positive definite. We have $\frac{1}{2}(\lambda_{\min}(\Gamma^{-1}) - \mu\lambda_S')\|\tilde{\theta}\|^2 \leq V_{\text{u}}(t,\tilde{\theta}) \leq \frac{1}{2}\lambda_{\min}(\Gamma^{-1})\|\tilde{\theta}\|^2$, where $\lambda_S' = \sup_{(t,x)\in\mathbb{R}_{\geq 0}\times K}\lambda_{\max}(S(t,x))$. It follows from this that V_{u} is positive definite provided that $\lambda_{\min}(\Gamma^{-1}) - \mu\lambda_S' > 0$, which is guaranteed if $\mu < \lambda_{\min}(\Gamma^{-1})/\lambda_S'$. When we insert $\hat{\phi} = \phi$, we get the same error dynamics as in the proof of Proposition 3. Following a calculation similar to the proof of Proposition 2, we get $\dot{V}_{\text{u}}(t,\tilde{\theta}) \leq -(1 - \frac{1}{2}\mu)\tilde{\theta}^{\mathsf{T}}S(t,x)\tilde{\theta} - \frac{1}{2}\mu\varepsilon\exp(-T)\|\tilde{\theta}\|^2 + \mu\sqrt{\kappa}M_S\|\Gamma\|M_M L_g\|\tilde{\theta}\|(\tilde{\theta}^{\mathsf{T}}S(t,x)\tilde{\theta})^{1/2}$, where M_S and M_M are bounds on $\|S(t,x)\|$ and $\|M(t,x,\hat{\theta})\|$ respectively, and κ is the ratio of the largest to the smallest eigenvalue of Γ^{-1}. We may write this as a quadratic expression with respect to $[(\tilde{\theta}^{\mathsf{T}}S(t,x)\tilde{\theta})^{1/2}, \|\tilde{\theta}\|]^{\mathsf{T}}$. It is then easily confirmed that the expression is negative definite if $\mu < 2/(1 + \kappa M_S^2\|\Gamma\|^2 M_M^2 L_g^2\varepsilon^{-1}\exp(T))$. ■

References

Annaswamy, A. M., Skantze, F. P., Loh, A.-P., 1998. Adaptive control of continuous time systems with convex/concave parametrization. Automatica 34 (1), 33–49.

Bakkeheim, J., Johansen, T. A., Smogeli, Ø. N., Sørensen, A. J., 2008. Lyapunov-based integrator resetting with application to marine thruster control. IEEE Trans. Contr. Syst. Tech. 16 (5), 908–917.

Bošković, J. D., 1995. Stable adaptive control of a class of first-order nonlinearly parameterized plants. IEEE Trans. Automat. Contr. 40 (2), 347–350.

Bošković, J. D., 1998. Adaptive control of a class of nonlinearly parameterized plants. IEEE Trans. Automat. Contr. 43 (7), 930–934.

Cao, C., Annaswamy, A. M., Kojić, A., 2003. Parameter convergence in nonlinearly parametrized systems. IEEE Trans. Automat. Contr. 48 (3), 397–412.

Chakrabortty, A., Arcak, M., 2009. Time-scale separation redesigns for stabilization and performance recovery of uncertain nonlinear systems. Automatica 45 (1), 34–44.

Fomin, V., Fradkov, A., Yakubovich, V. (Eds.), 1981. Adaptive Control of Dynamical Systems. Nauka, Moskow.

Grip, H. F., Saberi, A., Johansen, T. A., 2009. State and parameter estimation for linear systems with nonlinearly parameterized perturbations. In: Proc. 48th IEEE Conf. Dec. Contr. Shanghai, China.

Ioannou, P. A., Sun, J., 1996. Robust Adaptive Control. Prentice Hall, Upper Saddle River, NJ.

Khalil, H. K., 2001. Nonlinear Systems, 3rd Edition. Prentice Hall, Upper Saddle River, NJ.

Krstić, M., Kanellakopoulos, I., Kokotović, P. V., 1995. Nonlinear and Adaptive Control Design. Wiley, New York.

Lage, A. C. V. M., Frøyen, J., Sævareid, O., Fjelde, K. K., 2000. Underbalanced drilling dynamics: Two-phase flow modelling, experiments and numerical solutions techniques. In: Proc. Rio Oil & Gas Conf. Rio de Janeiro, Brazil.

Loh, A.-P., Annaswamy, A. M., Skantze, F. P., 1999. Adaptation in the presence of a general nonlinear parameterization: An error model approach. IEEE Trans. Automat. Contr. 44 (9), 1634–1652.

Loría, A., Panteley, E., Popović, D., Teel, A. R., 2005. A nested Matrosov theorem and persistency of excitation for uniform convergence in stable nonautonomous systems. IEEE Trans. Automat. Contr. 50 (2), 183–198.

Marino, R., Tomei, P., 1995. Nonlinear Control Design: Geometric, Adaptive and Robust. Prentice Hall, Upper Saddle River, NJ.

Nicosia, S., Tornambè, A., Valigi, P., 1994. Nonlinear map inversion via state observers. Circ. Syst. Signal Process. 13 (5), 571–589.

Nocedal, J., Wright, S. J., 1999. Numerical Optimization. Springer-Verlag, New York.

Nygaard, G., Imsland, L., Johannessen, E. A., 2007. Using NMPC based on a low-order model for controlling pressure during oil well drilling. In: Proc. 8th Int. Symp. on Dynamics and Control of Process Systems. Cancun, Mexico.

Nygaard, G., Nævdal, G., 2006. Nonlinear model predictive control scheme for stabilizing annulus pressure during oil well drilling. J. Process Contr. 16 (7), 719–732.

Ortega, R., 1996. Some remarks on adaptive neuro-fuzzy systems. Int. J. Adapt. Contr. Signal Process. 10 (1), 79–83.

Panteley, E., Loría, A., 2001. Growth rate conditions for uniform asymptotic stability of cascaded time-varying systems. Automatica 37 (3), 453–460.

Qu, Z., 2003. Adaptive and robust controls of uncertain systems with nonlinear parameterization. IEEE Trans. Automat. Contr. 48 (10), 1817–1823.

Qu, Z., Hull, R. A., Wang, J., 2006. Globally stabilizing adaptive control design for nonlinearly-parametrized systems. IEEE Trans. Automat. Contr. 51 (6), 1073–1079.

Reif, K., Günther, S., Yaz, E., Unbehauen, R., 1999. Stochastic stability of the discrete-time extended Kalman filter. IEEE Trans. Automat. Contr 44 (4), 714–728.

Stamnes, Ø. N., Zhou, J., Kaasa, G.-O., Aamo, O. M., 2008. Adaptive observer design for the bottomhole pressure of a managed pressure drilling system. In: Proc. 47th IEEE Conf. Dec. Contr. Cancun, Mexico.

Tyukin, I. Y., 2003. Adaptation algorithms in finite form for nonlinear dynamic objects. Automation and Remote Control 64 (6), 951–974.

Tyukin, I. Y., Prokhorov, D. V., van Leeuwen, C., 2007. Adaptation and parameter estimation in systems with unstable target dynamics and nonlinear parametrization. IEEE Trans. Automat.

Contr. 52 (9), 1543–1559.
Zhang, T., Ge, S. S., Hang, C. C., Chai, T. Y., 2000. Adaptive control of first-order systems with nonlinear parameterization. IEEE Trans. Automat. Contr. 45 (8), 1512–1516.