Output Synchronization for Heterogeneous Networks of Non-Introspective, Non-Right-Invertible Agents

Håvard Fjær Grip, Ali Saberi, Tao Yang, and Anton A. Stoorvogel

***Abstract*— We consider the output synchronization problem for a heterogeneous network of linear agents. The agents are non-introspective, meaning that they do not have access to their own state or output; the only information available to each agent comes from the network, in the form of a linear combination of its output relative to that of neighboring agents. The basis for our study is a design previously presented by the authors, which was based on an additional assumption requiring the agents to be right-invertible. Here we dispense with the assumption of right-invertibility and instead introduce a new assumption based on solvability of a particular set of regulator equations.**

I. Introduction

In recent years, a substantial amount of attention has been given to the *synchronization* problem, which involves making a network of agents agree asymptotically on their state or output trajectories. The challenge of this problem lies in the limited amount of information available to each agent—typically, a linear combination of its state or output relative to that of neighboring agents.

Much of the attention has been directed toward the problem of *state synchronization* in *homogeneous* networks—that is, networks where the agent models are identical—with each agent receiving information about its own state relative to that of neighboring agents [1]–[6]. Roy, Saberi, and Herlugson [7] and Yang, Roy, Wan, and Saberi [8] considered this type of problem for more general network topologies. Others have studied the case where the agents receive relative information about their own partial-state output [9]–[13]. In this context, Li, Duan, Chen, and Huang [12] introduced the idea of a distributed observer, which makes additional use of the network by allowing the agents to exchange information with their neighbors about their own internal estimates. Many of the results on the synchronization problem are rooted in the seminal work of Wu and Chua [14], [15].

A limited amount of research has also been conducted on *heterogeneous* networks—that is, networks where the agent models are non-identical [16]–[18]. In this case, the physical interpretation of one agent's internal state may be different from that of another agent, and it is often more meaningful to aim for *output synchronization*—that is, agreement on some partial-state output from each agent [19]–[23].

The heterogeneous designs mentioned above rely—explicitly or implicitly—on some sort of self-knowledge that is separate from the information transmitted over the network. In particular, the agents may be required to know their own state, their own output, or their own state/output relative to that of a reference trajectory. In recent papers [24], [25], the authors have introduced the term *introspective* agent to refer to agents that possess this type of self-knowledge, and focused instead on *non-introspective* agents—that is, agents that receive no information independent from the network. To the best of the authors' knowledge, the only work besides that of Grip, Yang, Saberi, and Stoorvogel [24], [25] that clearly applies to a well-defined class of heterogeneous networks with non-introspective agents is by Zhao, Hill, and Liu [26]. In their work, the only information available to each agent is a linear combination of outputs received over the network, but the agents are assumed to be passive.

A. Right-Invertibility

The networks considered by Grip et al. [24], [25] consist of n agents governed by models on the form

$$\dot{x}_i = A_i x_i + B_i u_i, \quad y_i = C_i x_i + D_i u_i, \tag{1}$$

where $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$, and $y_i \in \mathbb{R}^p$. Controllers are designed to achieve output synchronization—that is, $(y_i - y_j) \to 0$ for all $i, j \in \{1, \ldots, n\}$—under a set of assumptions about the network topology and the agents' dynamics. The most restrictive of these assumptions requires the dynamics between u_i and y_i, described by the quadruple (A_i, B_i, C_i, D_i), to be *right-invertible*.

Right-invertibility means that, given a reference output $\bar{y}(t)$ for $t \geq 0$, there exist an initial condition $x_i(0)$ and an input $u_i(t)$ for $t \geq 0$ such that $y_i(t) = \bar{y}(t)$ for all $t \geq 0$. A necessary (but not sufficient) condition for right-invertibility is that the dimension of the input u_i must be at least as large as the dimension of the output y_i, which is clearly restrictive.

B. Contribution of This Paper

In this paper we consider the output synchronization problem for heterogeneous networks of non-introspective agents using the same design approach as before [24], [25]. However, we dispense with the condition of right-invertibility

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The work of Håvard Fjær Grip was supported by the Research Council of Norway. The work of Ali Saberi and Tao Yang was partially supported by National Science Foundation grant NSF-0901137 and NAVY grants ONR KKK777SB001 and ONR KKK760SB0012.

by instead making an assumption regarding the solvability of a particular set of regulator equations. This allows us to achieve output synchronization for a large class of networks that could not previously be handled, and for which no design methodology exists in the current literature. We demonstrate this through a simulation example that includes ten agents with four different models, none of which are right-invertible. After this we investigate the implications of our new assumption in detail, to show that, even though right-invertibility is not required, solvability of the regulator equations is intrinsically tied to the properties of the *non-right-invertible dynamics* of each agent.

II. Problem Formulation

We consider a network of n agents on the form (1). The agents are non-introspective, meaning that agent i does not have access to its own output y_i. The only available information comes from the network, which provides each agent with a linear combination of its own output relative to that of the other agents. In particular, agent i has access to the quantity

$$\zeta_i = \sum_{j=1}^{n} a_{ij}(y_i - y_j),$$

where $a_{ij} \geq 0$ and $a_{ii} = 0$. The network can be described by a directed graph (digraph) $\mathscr{G}$ with nodes corresponding to the agents in the network and edges given by the coefficients a_{ij}. In particular, $a_{ij} > 0$ implies that an edge exists from agent j to i. Agent j is then called a *parent* of agent i, and agent i is called a *child* of agent j. The weight of the edge equals the magnitude of a_{ij}.

We shall make use of the matrix $G = [g_{ij}]$, where $g_{ii} = \sum_{j=1}^{n} a_{ij}$ and $g_{ij} = -a_{ij}$ for $j \neq i$. This matrix is known as the *Laplacian* matrix of the digraph $\mathscr{G}$ and has the property that the sum of the coefficients on each row is equal to zero. In terms of the coefficients of G, ζ_i can be rewritten as $\zeta_i = \sum_{j=1}^{n} g_{ij} y_j$.

In addition to ζ_i, we assume that the agents can exchange information about their internal estimates using the same network. Specifically, agent i is presumed to have access to the quantity

$$\hat{\zeta}_i = \sum_{j=1}^{n} a_{ij}(\eta_i - \eta_j) = \sum_{j=1}^{n} g_{ij}\eta_j,$$

where $\eta_j \in \mathbb{R}^p$ is a variable produced by agent j as part of the agreement protocol. This variable will be specified as part of the protocol design.

III. Protocol Design

To facilitate our design, we first make the following assumption.

Assumption 1: The digraph $\mathscr{G}$ has a directed spanning tree with root agent $K \in \{1,\dots,n\}$, such that for each $i \in \{1,\dots,n\} \setminus K$, (A_i, B_i) is stabilizable and (A_i, C_i) is observable.

Remark 1: A *directed tree* is a directed subgraph of $\mathscr{G}$, consisting of a subset of the nodes and edges, such that every node has exactly one parent, except a single root node with no parents. Furthermore, there must exist a directed path from the root to every other agent. A *directed spanning tree* is a directed tree that contains all the nodes of $\mathscr{G}$. A digraph may contain many directed spanning trees, and thus there may be several choices of root agent K.

In addition to Assumption 1, we need an assumption about the solvability of a particular set of regulator equations for each agent $i \in \{1,\dots,n\} \setminus K$. Because this assumption relies on a preliminary state transformation carried out during the design, we shall state it below as part of the design procedure.

The main idea behind the design is to let agent K operate autonomously by setting $u_K = 0$, and to also set $\eta_K = 0$. Controllers are then designed for all the other agents to make their outputs asymptotically synchronize with the trajectory $y_K(t)$. Thus, for each $i \in \{1,\dots,n\} \setminus K$, the objective is to regulate the synchronization error variable $e_i := y_i - y_K$ to zero. The controller design for each agent $i \in \{1,\dots,n\} \setminus K$ is described in the following section. The information needed by each agent in order to carry out this design is

- the matrices A_K and C_K
- a number n^* that is a bound on the order of any agent $i \in \{1,\dots,n\} \setminus K$
- a number $\tau > 0$ that is a lower bound on the real part of the eigenvalues of $\bar{G}$, where $\bar{G}$ is obtained by removing the K'th row and column from the Laplacian matrix G
- a common high-gain parameter $\varepsilon \in (0,1]$

Remark 2: Since $\mathscr{G}$ has a directed spanning tree, the eigenvalues of $\bar{G}$ all have strictly positive real parts [25, Lemma 4].

Remark 3: Without loss of generality, we can assume that (A_K, C_K) is observable and that all the eigenvalues of A_K are in the closed right-half complex plane (see [25] for details).

A. Design Procedure for Agent $i \in \{1,\dots,n\} \setminus K$

Define

$$O_i = \begin{bmatrix} C_i & -C_K \\ \vdots & \vdots \\ C_i A_i^{n_i+n_K-1} & -C_K A_K^{n_i+n_K-1} \end{bmatrix}. \tag{2}$$

Let q_i denote the dimension of the null space of O_i, and define $r_i = n_K - q_i$. Next, define $\Lambda_{iu} \in \mathbb{R}^{n_i \times q_i}$ and $\Phi_{iu} \in \mathbb{R}^{n_K \times q_i}$ such that $O_i \left[\begin{smallmatrix} \Lambda_{iu} \\ \Phi_{iu} \end{smallmatrix}\right] = 0$ and rank $\left[\begin{smallmatrix} \Lambda_{iu} \\ \Phi_{iu} \end{smallmatrix}\right] = q_i$. It is easy to show that Λ_{iu} and Φ_{iu} have full column rank (see [25]). Let Λ_{io} and Φ_{io} be defined such that $\Lambda_i := [\Lambda_{iu}, \Lambda_{io}] \in \mathbb{R}^{n_i \times n_i}$ and $\Phi_i := [\Phi_{iu}, \Phi_{io}] \in \mathbb{R}^{n_K \times n_K}$ are nonsingular.

Define a new state variable $\bar{x}_i \in \mathbb{R}^{n_i + r_i}$ as

$$\bar{x}_i = \begin{bmatrix} x_i - \Lambda_i M_i \Phi_i^{-1} x_K \\ -N_i \Phi_i^{-1} x_K \end{bmatrix}, \tag{3}$$

where $M_i = \left[\begin{smallmatrix} I_{q_i} & 0 \\ 0 & 0 \end{smallmatrix}\right] \in \mathbb{R}^{n_i \times n_K}$ and $N_i = \left[\begin{smallmatrix} 0 & I_{r_i} \end{smallmatrix}\right] \in \mathbb{R}^{r_i \times n_K}$.

From [25, Lemma 2], the synchronization error variable e_i is then governed by the equations

$$\dot{\bar{x}}_i = \bar{A}_i \bar{x}_i + \bar{B}_i u_i := \begin{bmatrix} A_i & \bar{A}_{i12} \\ 0 & \bar{A}_{i22} \end{bmatrix} \bar{x}_i + \begin{bmatrix} B_i \\ 0 \end{bmatrix} u_i, \tag{4a}$$

$$e_i = \bar{C}_i \bar{x}_i + \bar{D}_i u_i := \begin{bmatrix} C_i & -\bar{C}_{i2} \end{bmatrix} \bar{x}_i + D_i u_i, \tag{4b}$$

where $(\bar{A}_i, \bar{C}_i)$ is observable and the eigenvalues of $\bar{A}_{i22}$ are a subset of the eigenvalues of A_K. We can now state our additional assumption.

Assumption 2: There exist Π_i and Γ_i that solve the regulator equations

$$\Pi_i \bar{A}_{i22} = A_i \Pi_i + \bar{A}_{i12} + B_i \Gamma_i, \tag{5a}$$

$$C_i \Pi_i - \bar{C}_{i2} + D_i \Gamma_i = 0. \tag{5b}$$

Continuing the design, define $\bar{n} = n^* + n_K$ and $\chi_i = T_i \bar{x}_i$, where $T_i = [\bar{C}_i', \ldots, (\bar{C}_i \bar{A}_i^{\bar{n}-1})']'$. The matrix T_i is injective, and hence $T_i' T_i$ is invertible. In terms of the new state χ_i, we can write the system equations as

$$\dot{\chi}_i = (\mathscr{A} + \mathscr{L}_i)\chi_i + \mathscr{B}_i u_i, \quad e_i = \mathscr{C}\chi_i + \mathscr{D}_i u_i, \tag{6}$$

where

$$\mathscr{A} = \begin{bmatrix} 0 & I_{p(\bar{n}-1)} \\ 0 & 0 \end{bmatrix}, \quad \mathscr{C} = \begin{bmatrix} I_p & 0 \end{bmatrix},$$

$$\mathscr{L}_i = \begin{bmatrix} 0 \\ L_i \end{bmatrix}, \quad \mathscr{B}_i = T_i \begin{bmatrix} B_i \\ 0 \end{bmatrix}, \quad \mathscr{D}_i = D_i,$$

for a matrix $L_i \in \mathbb{R}^{p \times \bar{n}p}$ [25]. Let $\mathscr{P} = \mathscr{P}' > 0$ be the unique solution of the algebraic Riccati equation

$$\mathscr{A}\mathscr{P} + \mathscr{P}\mathscr{A}' - 2\tau \mathscr{P}\mathscr{C}'\mathscr{C}\mathscr{P} + I_{\bar{n}p} = 0, \tag{7}$$

and construct the observer

$$\dot{\hat{\chi}}_i = (\mathscr{A} + \mathscr{L}_i)\hat{\chi}_i + \mathscr{B}_i u_i + S(\varepsilon)\mathscr{P}\mathscr{C}'(\zeta_i - \hat{\zeta}_i), \tag{8a}$$

$$\hat{\bar{x}}_i = (T_i' T_i)^{-1} T_i' \hat{\chi}_i, \tag{8b}$$

where $S(\varepsilon) = \text{blkdiag}(I_p \varepsilon^{-1}, \ldots, I_p \varepsilon^{-\bar{n}})$ for a parameter $\varepsilon \in (0, 1]$. Finally, define $u_i = [F_i, \Gamma_i - F_i \Pi_i]\hat{\bar{x}}_i$, where F_i is chosen such that $A_i + B_i F_i$ is Hurwitz and Π_i and Γ_i are solutions of the regulator equations (5), and define $\eta_i = \mathscr{C}\hat{\chi}_i + \mathscr{D}_i u_i$.

The main result for this design can be summarized as follows.

Theorem 1: Suppose that Assumption 1 holds and that Assumption 2 holds for each $i \in \{1, \ldots, n\} \setminus K$. There exists an $\varepsilon^* \in (0, 1]$ such that, if $\varepsilon \in (0, \varepsilon^*]$, $i \in \{1, \ldots, n\} \setminus K$, then output synchronization is achieved.

Proof: The agreement protocol is the same as in our previous work [24], even though it is facilitated by different assumptions. Hence, the theorem follows trivially from our previous results. ∎

In the remainder of this paper it will be useful to have a good understanding of the first step of the design procedure, which involves a state transformation in order to describe e_i by (4). The purpose of this step is to reduce the dimension of the model underlying e_i by removing any redundant dynamics—specifically, dynamics in agent K that are contained within agent i. The pair $(\bar{A}_{i22}, \bar{C}_{i2})$ contains "what is left" after this reduction—that is, the dynamics of agent K that is not contained within agent i. To be more precise, we can state the following proposition.

Proposition 1: For some subspace $\mathscr{S}_i \subset \mathbb{R}^{n_K}$ of dimension ℓ_i, suppose that for each initial condition $x_K(0) \in \mathscr{S}_i$, there is an initial condition $x_i(0) \in \mathbb{R}^{n_i}$ such that, for $u_K = 0$ and $u_i = 0$, $y_i(t) = y_K(t)$ for all $t \geq 0$. Then $r_i \leq n_K - \ell_i$, where r_i is the dimension of the pair $(\bar{A}_{i22}, \bar{C}_{i2})$.

Proof: For any initial condition $x_K(0) \in \mathscr{S}_i$, we have $e_i = 0$ for some initial condition $x_i(0)$ if $u_K = 0$ and $u_i = 0$. It follows that these initial conditions are indistinguishable from the output e_i of the system $\dot{x}_i = A_i x_i$, $\dot{x}_K = A_K x_K$, $e_i = C_i x_i - C_K x_K$, and hence this system has an unobservable subspace of dimension greater than or equal to ℓ_i. The observability matrix of this system is O_i, and hence $q_i \geq \ell_i \implies r_i \leq n_K - \ell_i$. ∎

An immediate implication of Proposition 1 is that, if all open-loop solutions of agent K can be replicated by open-loop solutions of agent i, then $(\bar{A}_{i22}, \bar{C}_{i2})$ is of dimension zero.

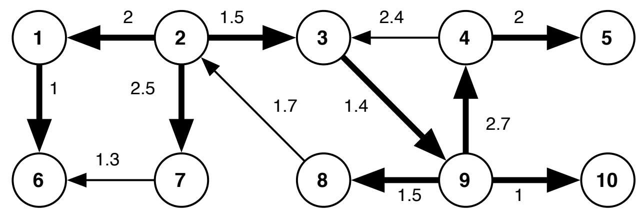

Fig. 1. Network graph

IV. EXAMPLE

In this section we consider an example network described by the graph depicted in Fig. 1. This graph contains multiple directed spanning trees. One of these is rooted at node 2, as illustrated by bold arrows. In our design we shall use $K = 2$, and it is therefore assumed that the other agents have knowledge of the pair (A_2, C_2). The matrix $\bar{G}$, obtained by removing row and column number 2 from the Laplacian of the network, has eigenvalues with real parts larger than approximately 0.33. We assume that the agents have knowledge of a lower bound $\tau = 0.3$. All the agents are of order 3, and we assume that the agents have knowledge of a bound $n^* = 3$.

The model of agents 1 and 2 is given by

$$A_i = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C_i = I.$$

The model of agent 3, 4, and 5 is given by

$$A_i = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C_i = I.$$

The model of agent 6, 7, and 8 is given by

$$A_i = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C_i = I.$$

Finally, the model of agents 9 and 10 is given by

$$A_i = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix}, \; B_i = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \; C_i = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

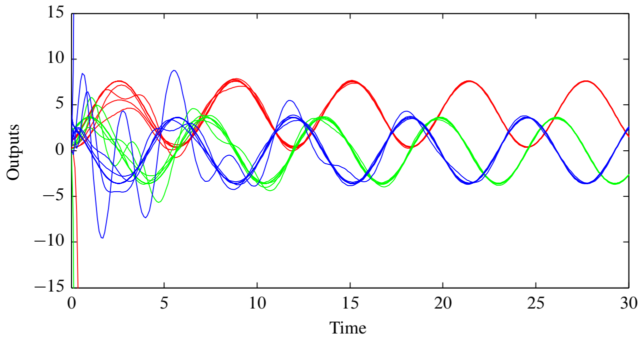

Fig. 2. Agent outputs for simulation example. The red lines show output number 1, the green lines show output number 2, and the blue lines show output number 3 for agents 1 through 10.

(We have $D_i = 0$ for all the agents.) Note that none of the agents are right-invertible. For illustrative purposes, we give the details of the design process for agent 3.

When we compute O_3 we find that $q_3 = 1$ and $r_3 = 2$, and that we may choose the matrices $\Lambda_{3u} = [1,0,0,]'$ and $\Phi_{3u} = [1,0,0]'$. Hence, one choice of Λ_3 and Φ_3 is

$$\Lambda_3 = I, \quad \Phi_3 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The dynamics of $\bar{x}_3$ with output e_3 then takes the form (4) with

$$\bar{A}_{312} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{A}_{322} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \bar{C}_{32} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We now attempt to solve the regulator equations (5), and find that they are solvable with

$$\Pi_3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Gamma_3 = \begin{bmatrix} -1 & 0 \end{bmatrix}.$$

The observer for $\bar{x}_3$ is designed following the procedure described in Section III-A with ε chosen as $\varepsilon = 0.3$, to provide an estimate of the transformed state $\bar{x}_3$. We select the matrix $F_3 = [-6,-11,-6]$ to place the poles of $A_3 + B_3F_3$ at -1, -2, and -3, which results in the control law $u_3 = [-6,-11,-6,10,0]\hat{\bar{x}}_3$. A similar procedure is carried out for the other agents as well. The simulated outputs from all the agents are shown in Fig. 2.

V. Solvability of the Regulator Equations

The above results should not be interpreted to mean that right-invertibility plays no role in determining whether output synchronization is achievable. In the following, we shall show that solvability of the regulator equations (5) depends fundamentally on the relationship between each agent's non-right-invertible dynamics and the pair $(\bar{A}_{i22}, \bar{C}_{i2})$.

It is easy to see from (4) that the task of ensuring $e_i \to 0$ is equivalent to regulating the output of a system

$$\dot{\bar{x}}_{i1} = A_i\bar{x}_{i1} + \bar{A}_{i12}\bar{x}_{i2} + B_iu_i, \quad \bar{y}_{i1} = C_i\bar{x}_{i1} + D_iu_i, \tag{9}$$

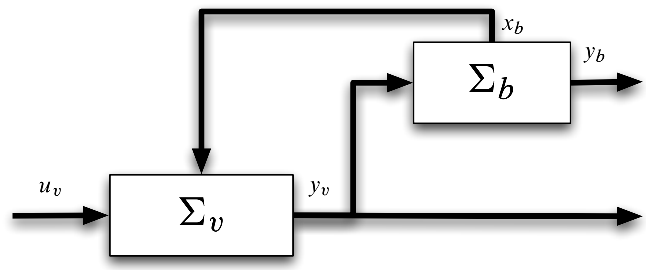

Fig. 3. System partitioned into right-invertible and non-right-invertible dynamics

to the output of an exosystem

$$\dot{\bar{x}}_{i2} = \bar{A}_{i22}\bar{x}_{i2}, \quad \bar{y}_{i2} = \bar{C}_{i2}\bar{x}_{i2}. \tag{10a}$$

Solvability of the regulator equations (5) is closely related to the ability of (9) to reproduce the output from (10) for any initial condition. In particular, solvability ensures that there exists a state-feedback control law $u_i = F_i\bar{x}_{i1} + (\Gamma_i - F_i\Pi_i)\bar{x}_{i2}$ that achieves the regulation task. To see this, define a variable $\tilde{x}_i = \bar{x}_{i1} - \Pi_i\bar{x}_{i2}$, and note that by using the regulator equations (5), we can write

$$\begin{aligned}\dot{\tilde{x}}_i &= A_i\bar{x}_{i1} + \bar{A}_{i12}\bar{x}_{i2} + B_iu_i - \Pi_i\bar{A}_{i22}\bar{x}_{i2} \\ &= A_i(\bar{x}_{i1} - \Pi_i\bar{x}_{i2}) + B_iF_i(\bar{x}_{i1} - \Pi_i\bar{x}_{i2}) = (A_i + B_iF_i)\tilde{x}_i.\end{aligned}$$

Thus, with F_i chosen such that $A_i + B_iF_i$ is Hurwitz, $\tilde{x}_i \to 0$. Furthermore, we have

$$\begin{aligned}e_i &= C_i\bar{x}_{i1} - \bar{C}_{i2}\bar{x}_{i2} + D_iu_i \\ &= C_i(\bar{x}_{i1} - \Pi_i\bar{x}_{i2}) + D_iF_i(\bar{x}_{i1} - \Pi_i\bar{x}_{i2}) = (C_i + D_iF_i)\tilde{x}_i,\end{aligned}$$

and hence $e_i \to 0$.

A. Right-Invertible and Non-Right-Invertible Dynamics

In order to study solvability conditions for the regulator equations, we shall make use of a canonical form that separates a system into right-invertible and non-right-invertible dynamics.[1] Given an arbitrary linear time-invariant system

$$\dot{x} = Ax + Bu, \quad y = Cx + Du,$$

there exist nonsingular state, input, and output transformations Ψ_x, Ψ_u, and Ψ_y such that, by defining $x = \Psi_x[x_b', x_v']'$, $u = \Psi_u u_v$, and $y = \Psi_y[y_b', y_v']'$, the system can be described in terms of two subsystems:

$$\Sigma_b : \begin{cases} \dot{x}_b = A_bx_b + L_{bv}y_v, \\ y_b = C_bx_b, \end{cases} \tag{11a}$$

$$\Sigma_v : \begin{cases} \dot{x}_v = A_vx_v + B_vu_v + E_{vb}x_b, \\ y_v = C_vx_v + D_vu_v + H_{vb}x_b. \end{cases} \tag{11b}$$

Fig. 3 illustrates how these two subsystems are connected. We see that Σ_b is not directly influenced by the input u_v;

[1]The canonical form can be obtained by transforming the system to the *special coordinate basis* [27] and combining the states x_a, x_c, and x_d into a single state x_v; the inputs u_0, u_c and u_d into a single input u_v; and the outputs y_0 and y_d into a single output y_v.

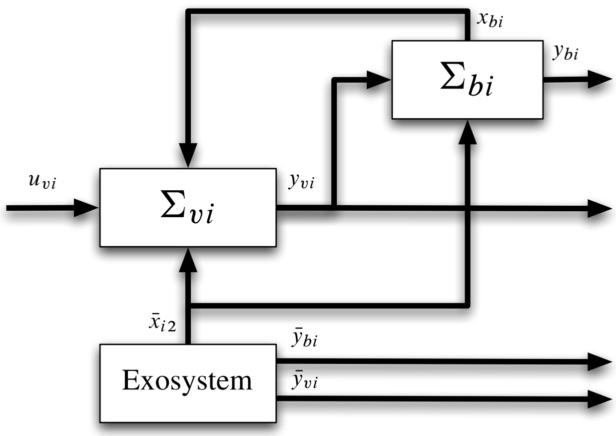

Fig. 4. System partitioned into right-invertible and non-right-invertible dynamics

it is only indirectly influenced via the output y_v from Σ_v. Hence, y_b is entirely dictated by y_v together with the initial condition $x_b(0)$.

The Σ_b subsystem is said to represent the system's non-right-invertible dynamics. To see why, note that right-invertibility would require the ability of y_b and y_v to track arbitrary reference signals $\bar{y}_b(t)$ and $\bar{y}_v(t)$ by the proper choice of initial conditions and input u_v. However, if $y_v(t) = \bar{y}_v(t)$, then there is no freedom left in the input to Σ_b to ensure that $y_b(t) = \bar{y}_b(t)$. Right-invertibility can therefore only occur if x_b is of dimension zero. On the other hand, it is a property of the canonical form that (A_v, B_v, C_v, D_v) is right-invertible; thus, the system is right-invertible in the absence of Σ_b. The Σ_v subsystem is therefore said to represent the right-invertible dynamics. We also note that (A_v, B_v, C_v, D_v) has the same invariant zeros as (A, B, C, D) and that (A_b, C_b) is observable.

B. Non-Right-Invertible Dynamics and the Regulator Equations

Let us now turn to the problem regulating the output of (9) to the output of (10). If we apply state, input, and output transformations $\bar{x}_{i1} = \Psi_{xi}[x'_{bi}, x'_{vi}]'$, $u_i = \Psi_{ui} u_{vi}$, and $\bar{y}_{i1} = \Psi_{yi}[y'_{bi}, y'_{vi}]'$ to the system (9) in order to bring it into the canonical form, we obtain the system equations

$$\Sigma_{bi} : \begin{cases} \dot{x}_{bi} = A_{bi} x_{bi} + L_{bvi} y_{vi} + G_{bi} \bar{x}_{i2}, \\ y_{bi} = C_{bi} x_{bi}, \end{cases} \tag{12a}$$

$$\Sigma_{vi} : \begin{cases} \dot{x}_{vi} = A_{vi} x_{vi} + B_{vi} u_{vi} + E_{vbi} x_{bi} + G_{vi} \bar{x}_{i2}, \\ y_{vi} = C_{vi} x_{vi} + D_{vi} u_{vi} + H_{vbi} x_{bi}, \end{cases} \tag{12b}$$

where the terms $G_{bi}\bar{x}_{i2}$ and $G_{vi}\bar{x}_{i2}$ are due to the term $\bar{A}_{i12}\bar{x}_{i2}$ in (9). We denote by n_{bi} and n_{vi} the dimensions of the Σ_{bi} and Σ_{vi} subsystems. Applying the same output transformation $\bar{y}_{i2} = \Psi_{yi}[\bar{y}'_{bi}, \bar{y}'_{vi}]'$ to the exosystem (10), we can write it as

$$\dot{\bar{x}}_{i2} = \bar{A}_{i22} \bar{x}_{i2}, \quad \bar{y}_{bi} = \bar{C}_{bi} \bar{x}_{i2}, \quad \bar{y}_{vi} = \bar{C}_{vi} \bar{x}_{i2}. \tag{13}$$

The goal is now to ensure $(y_{bi} - \bar{y}_{bi}) \to 0$ and $(y_{vi} - \bar{y}_{vi}) \to 0$.

Fig. 4 shows how the systems (12) and (13) are interconnected. Analogous to our discussion in Section V-A, it is evident that if $y_{vi}(t) = \bar{y}_{vi}(t)$, then there is no freedom left to force y_{bi} to track $\bar{y}_{bi}(t)$. Thus, whether the regulation problem can be solved depends inherently on the relationship between the non-right-invertible dynamics and the dynamics of the exosystem.

Theorem 2: Define $\bar{O}_i$ as the observability matrix of the pair

$$\left(\begin{bmatrix} A_{bi} & L_{bvi}\bar{C}_{vi} + G_{bi} \\ 0 & \bar{A}_{i22} \end{bmatrix}, \begin{bmatrix} C_{bi} & -\bar{C}_{bi} \end{bmatrix} \right).$$

A necessary condition for solvability of the regulator equations (5) is that $\operatorname{rank} \bar{O}_i = n_{bi}$. A sufficient condition for solvability of the regulator equations (5) is that $\operatorname{rank} \bar{O}_i = n_{bi}$ and, additionally, that (A, B, C, D) has no invariant zeros coinciding with the eigenvalues of $\bar{A}_{i22}$.

Proof: We first prove the sufficiency part. Considering the system equations in (12) and (13), we find that

$$\Psi_{xi}^{-1} A_i \Psi_{xi} = \begin{bmatrix} A_{bi} + L_{bvi} H_{vbi} & L_{bvi} C_{vi} \\ E_{vbi} & A_{vi} \end{bmatrix},$$

$$\Psi_{xi}^{-1} B_i \Psi_{ui} = \begin{bmatrix} L_{bvi} D_{vi} \\ B_{vi} \end{bmatrix}, \; \Psi_{yi}^{-1} C_i \Psi_{xi} = \begin{bmatrix} C_{bi} & 0 \\ H_{vbi} & C_{vi} \end{bmatrix},$$

$$\Psi_{yi}^{-1} D_i \Psi_{ui} = \begin{bmatrix} 0 \\ D_{vi} \end{bmatrix}, \Psi_{xi}^{-1} \bar{A}_{i12} = \begin{bmatrix} G_{bi} \\ G_{vi} \end{bmatrix}, \Psi_{yi}^{-1} \bar{C}_{i2} = \begin{bmatrix} \bar{C}_{bi} \\ \bar{C}_{vi} \end{bmatrix}.$$

Using these expressions, it is easy to verify that (5) is equivalent to

$$\begin{bmatrix} \Pi_{bi} \\ \Pi_{vi} \end{bmatrix} \bar{A}_{i22} = \begin{bmatrix} A_{bi} + L_{bvi} H_{vbi} & L_{bvi} C_{vi} \\ E_{vbi} & A_{vi} \end{bmatrix} \begin{bmatrix} \Pi_{bi} \\ \Pi_{vi} \end{bmatrix} + \begin{bmatrix} G_{bi} \\ G_{vi} \end{bmatrix} + \begin{bmatrix} L_{bvi} D_{vi} \\ B_{vi} \end{bmatrix} \Gamma_{vi}, \tag{14a}$$

$$\begin{bmatrix} C_{bi} & 0 \\ H_{vbi} & C_{vi} \end{bmatrix} \begin{bmatrix} \Pi_{bi} \\ \Pi_{vi} \end{bmatrix} - \begin{bmatrix} \bar{C}_{bi} \\ \bar{C}_{vi} \end{bmatrix} + \begin{bmatrix} 0 \\ D_{vi} \end{bmatrix} \Gamma_{v_i} = 0, \tag{14b}$$

with $\Pi_i = \Psi_{xi}[\Pi'_{bi}, \Pi'_{vi}]'$ and $\Gamma_i = \Psi_{ui}\Gamma_{vi}$. We can write $\bar{O}_i = [\bar{O}_{i1}, \bar{O}_{i2}]$, where $\bar{O}_{i1} = [C'_{bi}, \ldots, (C_{bi} A_{bi}^{n_{bi}+r_i-1})']'$. Since (A_{bi}, C_{bi}) is observable, $\operatorname{rank} \bar{O}_{i1} = n_{bi}$. Since $\operatorname{rank} \bar{O}_i = n_{bi}$, we must have $\operatorname{im} \bar{O}_{i2} \subset \operatorname{im} \bar{O}_i = \operatorname{im} \bar{O}_{i1}$. Hence, there exists a Π_{bi} such that $\bar{O}_{i1}\Pi_{bi} + \bar{O}_{i2} = 0$, which means that $[\Pi'_{bi}, I]'$ is a basis for the (invariant) unobservable subspace of the matrix pair in the theorem. It follows that there is a V_i such that

$$\begin{bmatrix} A_{bi} & L_{bvi}\bar{C}_{vi} + G_{bi} \\ 0 & \bar{A}_{i22} \end{bmatrix} \begin{bmatrix} \Pi_{bi} \\ I \end{bmatrix} = \begin{bmatrix} \Pi_{bi} \\ I \end{bmatrix} V_i, \tag{15a}$$

$$\begin{bmatrix} C_{bi} & -\bar{C}_{bi} \end{bmatrix} \begin{bmatrix} \Pi_{bi} \\ I \end{bmatrix} = 0. \tag{15b}$$

It is obvious that we must have $V_i = \bar{A}_{i22}$ for this to hold, and it then follows that

$$\Pi_{bi} \bar{A}_{i22} = A_{bi} \Pi_{bi} + L_{bvi} \bar{C}_{vi} + G_{bi}. \tag{16}$$

Consider now the regulator equations

$$\Pi_{vi} \bar{A}_{i22} = A_{vi} \Pi_{vi} + E_{vbi} \Pi_{bi} + G_{vi} + B_{vi} \Gamma_{vi}, \tag{17a}$$

$$C_{vi} \Pi_{vi} + H_{vbi} \Pi_{bi} - \bar{C}_{vi} + D_{vi} \Gamma_{vi} = 0, \tag{17b}$$

where Π_{vi} and Γ_{vi} are the unknowns. Because $(A_{vi}, B_{vi}, C_{vi}, D_{vi})$ is right-invertible, the Rosenbrock system matrix $\begin{bmatrix} A_{vi}-\lambda I & B_{vi} \\ C_{vi} & D_{vi} \end{bmatrix}$ has normal rank $n_{vi}+p$ (see [28, Property 3.1.6]). Since no invariant zeros of $(A_{vi}, B_{vi}, C_{vi}, D_{vi})$ coincide with eigenvalues of $\bar{A}_{i22}$, the matrix retains its normal rank for all λ that are eigenvalues of $\bar{A}_{i22}$. It therefore follows that the regulator equations (17) are solvable [29, Corollary 2.5.1]. From (17), we see that $\bar{C}_{vi} = C_{vi}\Pi_{vi} + H_{vbi}\Pi_{bi} + D_{vi}\Gamma_{vi}$. Inserting this into (16), we have

$$\begin{aligned} \Pi_{bi}\bar{A}_{i22} = (A_{bi} + L_{bvi}H_{vbi})\Pi_{bi} + L_{bvi}C_{vi}\Pi_{vi} \\ + L_{bvi}D_{vi}\Gamma_{vi} + G_{bi}. \end{aligned} \tag{18}$$

Combining (17), (18), and the expression $C_{bi}\Pi_{bi} - \bar{C}_{bi} = 0$ from (15b), we see that Π_{bi}, Π_{vi}, and Γ_{vi} are solutions to (14), and hence the regulator equations (5) are solvable.

To prove the necessity part, note that on the agreement manifold, we must have $y_{vi} = \bar{y}_{vi}$, which means that on this manifold the difference between y_{bi} and $\bar{y}_{bi}$ is governed by

$$\begin{aligned} &\dot{x}_{bi} = A_{bi}x_{bi} + (L_{bvi}\bar{C}_{vi} + G_{bi})\bar{x}_{i2}, \quad \dot{\bar{x}}_{i2} = \bar{A}_{i22}\bar{x}_{i2}, \\ &y_{bi} - \bar{y}_{bi} = C_{bi}x_{bi} - \bar{C}_{bi}\bar{x}_{i2}. \end{aligned}$$

This dynamics corresponds precisely to the matrix pair in the statement of the theorem. In order to have $y_{bi} - \bar{y}_{bi} = 0$, we must therefore have $\bar{O}_{i1}x_{bi} + \bar{O}_{i2}\bar{x}_{i2} = 0$. If $\operatorname{rank}\bar{O}_i > \operatorname{rank}\bar{O}_{i1} = n_{bi}$, then this expression can only be satisfied for $\bar{x}_{i2}$ in some subspace of dimension lower than r_i, which means that $\bar{x}_{i2}$ must converge to this subspace. However, since the poles of $\bar{A}_{i22}$ are all in the closed right-half plane, there is no lower-dimensional subspace to which all solutions converge, and hence we must have $\operatorname{rank}\bar{O}_i = n_{bi}$. ■

The following corollary follows immediately from Theorem 2 and Proposition 1.

Corollary 1: The regulator equations (5) are always solvable if either (i) agent i is right-invertible and has no invariant zeros coinciding with the poles of $\bar{A}_{i22}$; or (ii) for each initial condition $x_K(0) \in \mathbb{R}^{n_K}$ there exists an initial condition $x_i(0) \in \mathbb{R}^{n_i}$ such that, for $u_i = 0$ and $u_K = 0$, $x_i(t) = x_K(t)$ for all $t \geq 0$.

VI. Concluding Remarks

In this paper we have shown that we can apply a previously developed design methodology for output synchronization in heterogeneous networks of non-introspective agents while dispensing with the assumption of right-invertibility. As a result, the class of networks for which output synchronization can be achieved is significantly expanded.

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