

Output Synchronization for Heterogeneous Networks of Introspective Right-Invertible Agents

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SUMMARY

In this paper, we consider the output synchronization problem for heterogeneous networks of right-invertible linear agents. We assume that all the agents are *introspective*, meaning that they have access to their own local measurements. Under this assumption, we then propose a decentralized control scheme for solving the output synchronization problem for a set of network topologies. The proposed scheme can also be applied to solve the output formation problem with arbitrary formation vectors. We also consider *the regulation of output synchronization* problem, where the output of each agent has to track an *a priori* specified reference trajectory, generated by an exosystem. In this case, we assume that the root agent has access to its own output relative to the reference trajectory. Copyright © 2012 John Wiley & Sons, Ltd.

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1. INTRODUCTION

The synchronization problem in a network has received substantial attention in recent years (see [9, 31, 18, 1] and references therein). Active research is being conducted in this context and numerous results have been reported in the literature, to name a few see [10, 11, 14, 15, 17, 24, 12, 25, 8, 23].

Much of the attention has been devoted to achieving *state synchronization* in *homogeneous* networks (i.e., networks where the agent models are identical), where each agent has access to a linear combination of its own state relative to that of neighboring agents (e.g., [10, 11, 9, 17, 15, 24]). Roy, Saberi, and Herlugson [19], Tuna [24], and Yang, Roy, Wan, and Saberi [33] considered the state synchronization problem for more general network topologies. A more realistic scenario—that is, each agent receives a linear combination of its own partial-state output relative to that of neighboring agents—has been considered in [13, 25, 26, 8]. The results of [8] were expanded by [34] to more general network topologies. Many of the results on the synchronization problem are rooted in the seminal work of Wu and Chua [29, 30].

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1.1. Heterogeneous networks and output synchronization

Recent activities in the synchronization literature have been focused on achieving synchronization in *heterogeneous* networks (i.e., networks where the agent models are non-identical). This problem is challenging and only some partial results are available, see for instance [6, 32, 4, 7, 28].

In heterogeneous networks, the agents' states may have different dimensions. In this case, the state synchronization is not even properly defined, and it is more natural to aim for *output synchronization*—that is, asymptotic agreement on the agents' partial-state outputs. Chopra and Spong [4] studied the output synchronization problem for weakly minimum-phase nonlinear systems of relative degree one, using a pre-feedback to create a single-integrator system with decoupled zero dynamics. Kim, Shim, and Seo [7] considered the output synchronization problem for uncertain single-input single-output, minimum-phase linear systems, by embedding an identical model within each agent, the output of which is tracked by the actual agent output.

The designs mentioned in this section generally rely on some sort of self-knowledge that is separate from the information transmitted over the network. More specifically, the agents know their own state, their own output, or their own state/output relative to that of the reference trajectory. We shall refer to agents that possess this type of self-knowledge as *introspective* agents, to distinguish them from *non-introspective* agents – that is, agents that have no knowledge of their own state or output separate from what is received via the network. The output synchronization problem for a heterogeneous network of *non-introspective* agents have been considered in [5] and [35].

1.2. Organization of this paper

The remainder of this paper is organized as follows. In this rest of Section 1, we introduce some notations and recall some results of algebraic graph theory. Section 2 presents the heterogeneous network considered in this paper. In Section 3, we propose a decentralized controller to solve the output synchronization problem. The design is applied for solving the output formation problem in Section 4. The regulation of output synchronization problem is considered in Section 5. The results are illustrated by examples in Section 6.

1.3. Preliminaries and notations

Given a matrix $A \in \mathbb{C}^{m \times n}$, A^* denotes its conjugate transpose, and $\lambda_i(A)$ is its i 'th eigenvalue. $A \in \mathbb{C}^{n \times n}$ is said to be Hurwitz stable if all its eigenvalues are in the open left-half plane. $\otimes$ denotes the Kronecker product between two matrices of appropriate dimensions. Given a matrix $A \in \mathbb{C}^{m \times n}$ and a matrix $B \in \mathbb{C}^{p \times q}$ the Kronecker product $A \otimes B$ is defined as the $\mathbb{C}^{mp \times nq}$ matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{bmatrix},$$

where a_{ij} denotes element (i, j) of A . I_n denotes the identity matrix of dimension n , similarly, 0_n denotes the square matrix of dimension n with all zero elements; we sometimes drop the subscript if the dimension is clear in the context. $\mathbf{1}$ denotes a column vector with all entries equal to one whose dimension should be clear from the context. For column vectors $x_1, \dots, x_n$, $[x_1; \dots; x_n]$ denotes the column vector by stacking the elements of $x_1, \dots, x_n$.

2. HETEROGENEOUS NETWORK

Consider a heterogeneous network of N linear agents

$$\begin{cases} \dot{x}_i = A_i x_i + B_i u_i, \\ y_i = C_i x_i \end{cases} \quad (1)$$

for $i \in \{1, \dots, N\}$, where $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$, $y_i \in \mathbb{R}^p$.

The agents are introspective, meaning that the agents have access to their own local information. Specifically, each agent has access to the quantity

$$z_i = C_i^m x_i. \quad (2)$$

where $z_i \in \mathbb{R}^{q_i}$.

The network infrastructure provides each agent with a linear combination of its own output relative to that of other agents. In particular, each agent i has access to the quantity

$$\zeta_i = \sum_{j=1}^N a_{ij}(y_i - y_j), \quad (3)$$

where $a_{ij} \geq 0$ and $a_{ii} = 0$ with $i, j \in \{1, \dots, n\}$. This network can be described by a weighted directed graph (digraph) $\mathcal{G}$ with nodes corresponding to the agents in the network and edges with weight given by the coefficients a_{ij} . In particular, $a_{ij} > 0$ means that there exists an edge with weight a_{ij} from agent j to agent i .

We also define a matrix $G = [g_{ij}]$, where $g_{ii} = \sum_{j=1}^N a_{ij}$ and $g_{ij} = -a_{ij}$ for $j \neq i$. The matrix G , known as the *weighted Laplacian* matrix of the digraph $\mathcal{G}$ has the property that the sum of the coefficients on each row is equal to zero. In terms of the coefficients g_{ij} of G , ζ_i given by (3) can be rewritten as

$$\zeta_i = \sum_{j=1}^N g_{ij} y_j. \quad (4)$$

With the local information z_i given by (2) and the information ζ_i given by (4) provided by the network, the agent i , where $i \in \{1, \dots, N\}$, has the following dynamical equations:

$$\begin{cases} \dot{x}_i = A_i x_i + B_i u_i, \\ y_i = C_i x_i, \\ z_i = C_i^m x_i, \\ \zeta_i = \sum_{j=1}^N g_{ij} y_j. \end{cases} \quad (5)$$

We make the following assumption regarding the network communication topology:

Assumption 1

The digraph $\mathcal{G}$ has a directed spanning tree.

From [16, Lemma 3.3], it is well known that under Assumption 1, the weighted Laplacian matrix G associated with network topology $\mathcal{G}$ has a simple eigenvalue at the origin, with the corresponding right eigenvector $\mathbf{1}$, and all the other eigenvalues are in the open right-half complex plane. We then let $\lambda_1, \dots, \lambda_N$ denote the eigenvalues of G , such that $\lambda_1 = 0$ and $0 < \text{Re}(\lambda_2) \leq \dots \leq \text{Re}(\lambda_N)$.

Let us now introduce the following definition to characterize a set of network communication topologies:

Definition 1

For any given $\gamma \geq \beta > 0$, let $\Gamma_{\beta, \gamma}$ denote the set of digraphs that satisfy Assumption 1 and for which the corresponding Laplacian matrix has the following properties: $\text{Re}(\lambda_2) \geq \beta$, and $\max_{i=2, \dots, N} |\lambda_i| < \gamma$ for $i \in \{2, \dots, N\}$.

Assumption 2

For each agent $i \in \{1, \dots, N\}$, we make the following assumption:

1. (A_i, B_i) is stabilizable;
2. (C_i, A_i) is detectable;
3. (C_i, A_i, B_i) is right-invertible; and
4. (C_i^m, A_i) is detectable.

Remark 1

Right-invertibility of a triple (C_i, A_i, B_i) means that, given a reference output $y_r(t)$, there exist an initial condition $x_i(0)$ and an input $u_i(t)$ such that $y_i(t) = y_r(t)$ for all $t \geq 0$. For example, every single-input single-output system is right-invertible, unless its transfer function is identically zero.

3. OUTPUT SYNCHRONIZATION

In this section, we consider the output synchronization problem for a heterogeneous network. The output synchronization is defined as follows:

Definition 2

A heterogeneous network of N agents is said to achieve output synchronization if

$$\lim_{t \rightarrow \infty} (y_i(t) - y_j(t)) = 0, \quad \forall i, j \in \{1, \dots, N\}.$$

Let us now formally formulate the output synchronization problem for a heterogeneous network.

Problem 1 (Output Synchronization)

Consider a heterogeneous network of N agents (5). For any given $\gamma \geq \beta > 0$, and the resulting set $\Gamma_{\beta, \gamma}$ of communication topologies, the output synchronization problem is to find, if possible, a linear dynamical controller

$$\begin{cases} \dot{x}_{i,c} = A_{i,c}x_{i,c} + B_{i,c}\zeta_i + E_{i,c}z_i, \\ u_i = C_{i,c}x_{i,c} + D_{i,c}\zeta_i + M_{i,c}z_i, \end{cases} \quad (6)$$

for each agent $i \in \{1, \dots, N\}$, such that output synchronization is achieved for any network communication topology represented by the digraph $\mathcal{G} \in \Gamma_{\beta, \gamma}$.

Remark 2

Since (C_i^m, A_i) is detectable for $i \in \{1, \dots, N\}$, one can, without any communication among agents, simply asymptotically stabilize each individual agent by utilizing z_i , and hence achieve the output synchronization with zero synchronization trajectory, that is $\lim_{t \rightarrow \infty} y_i(t) = 0$, $i \in \{1, \dots, N\}$. In this paper, we are not interested in such a case. We are aiming to achieve output synchronization with non-trivial synchronization trajectories.

Theorem 1

Consider a heterogeneous network of N agents (5). Let Assumptions 1 and 2 hold. Then the output synchronization problem with $\Gamma_{\beta, \gamma}$ for any $\gamma \geq \beta > 0$ as defined in Problem 1, is solvable via N decentralized controllers of the form (6).

We shall prove Theorem 1 by explicit construction of synchronization controllers for each agent. The fundamental challenge of the output synchronization problem for heterogeneous networks is that the agent models are non-identical. Therefore, we first design a local pre-compensator to make all the agents almost identical, which we refer to as *homogenization of network*. Next, we show that the output synchronization problem with respect to the new almost identical models can be converted into a simultaneous stabilization problem. Finally, we design controllers via a low-gain approach to solve the reformulated simultaneous stabilization problem in the homogenized network.

3.1. Homogenization of network

Since each agent is introspective, we use the local information z_i to manipulate the agent dynamics such that all the agents' models are almost identical to the rest of network. This is shown in the following lemma.

Lemma 1

Consider a heterogeneous network of N agents (5). Let Assumption 2 hold, and let $\bar{n}_d$ denote the maximal order of infinite zeros of (C_i, A_i, B_i) , $i \in \{1, \dots, N\}$. Suppose a triple (C, A, B) is given such that

1. $\text{rank}(C) = p$,
2. (C, A, B) is invertible, of uniform rank $n_q \geq \bar{n}_d$, and has no invariant zeros.

Then for each agent $i \in \{1, \dots, N\}$, there exist a pre-compensator of the form

$$\begin{cases} \dot{\xi}_i = A_{i,h}\xi_i + B_{i,h}z_i + E_{i,h}v_i, \\ u_i = C_{i,h}\xi_i + D_{i,h}v_i, \end{cases} \quad (7)$$

such that the interconnection of (5) and (7) can be written in the following form:

$$\begin{cases} \dot{\bar{x}}_i = A\bar{x}_i + B(v_i + \rho_i), \\ y_i = C\bar{x}_i, \\ \zeta_i = \sum_{j=1}^N g_{ij}y_j, \end{cases} \quad (8)$$

where ρ_i is given by

$$\begin{cases} \dot{\omega}_i = A_{i,s}\omega_i, \\ \rho_i = C_{i,s}\omega_i, \end{cases} \quad (9)$$

and $A_{i,s}$ is Hurwitz stable.

Proof

The proof of Lemma 1 will be given in Appendix B by explicit construction of a pre-compensator of the form (7). $\square$

Remark 3

We would like to make several observations:

1. The property that the triple (C, A, B) is invertible and has no invariant zero implies that (A, B) is controllable and (C, A) is observable.
2. The triple (C, A, B) is arbitrarily assignable as long as the conditions are satisfied. They play a role as design parameters. We shall use this freedom in various places in this paper.

Remark 4

Without loss of generality, we assume that the triple (C, A, B) has the following form:[†]

$$A = A_0 + BH, \quad A_0 := \begin{bmatrix} 0 & I_{p(n_q-1)} \\ 0 & 0 \end{bmatrix}, \quad B = B_0 := \begin{bmatrix} 0 \\ I_p \end{bmatrix}, \quad C = C_0 := [I_p \quad 0], \quad (10)$$

where H is such that the matrix $A_0 + B_0H$ has desired eigenvalues. The existence of such an H is guaranteed by the fact that (A_0, B_0) is controllable.

Lemma 1 shows that we can design a pre-compensator based on local information z_i to transform each non-identical agent model given by (5) into a new model given by (8) and (9). The new agent models (8) are almost identical except for different exponentially decaying signals ρ_i in the range space of B , generated by (9). We shall solve the output synchronization problem with respect to the new almost identical models (8) and (9), and then combine the result with Lemma 1 to prove Theorem 1.

3.2. Connection to simultaneous stabilization problem

In this section, we show that the exponentially decaying signals ρ_i are irrelevant for solving the output synchronization problem with respect to the new almost identical models (8) and (9), and that the problem is essentially reduced to a simultaneous stabilization problem.

For solving the synchronization problem for a network of N agents (8) and (9) with a set of possible communication topologies $\Gamma_{\beta,\gamma}$, we consider N general decentralized controllers of the form (11)

$$\begin{cases} \dot{\chi}_i = A_k\chi_i + B_k\zeta_i, \\ v_i = C_k\chi_i, \end{cases} \quad (11)$$

for $i \in \{1, \dots, N\}$, where $\chi_i \in \mathbb{R}^{n_c}$, which should be independent of the specific communication topology $\mathcal{G} \in \Gamma_{\beta,\gamma}$.

[†]If (C, A, B) is not in this form, from [22], which is also reviewed in Appendix A.1, there exist nonsingular state and input transformations, such that the transformed system is in this form.

With $\tilde{x}_i := [\bar{x}_i; \chi_i]$, the closed-loop system of (8) and (11) for each individual agent can be written as

$$\begin{cases} \dot{\tilde{x}}_i = \begin{bmatrix} A & BC_k \\ 0 & A_k \end{bmatrix} \tilde{x}_i + \begin{bmatrix} 0 \\ B_k \end{bmatrix} \zeta_i + \begin{bmatrix} B \\ 0 \end{bmatrix} \rho_i, \\ y_i = [C \ 0] \tilde{x}_i, \\ \zeta_i = \sum_{j=1}^N g_{ij} y_j. \end{cases} \quad (12)$$

Define $\tilde{x} := [\tilde{x}_1; \dots; \tilde{x}_N]$, $\rho := [\rho_1; \dots; \rho_N]$,

$$\bar{A} = \begin{bmatrix} A & BC_k \\ 0 & A_k \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 \\ B_k \end{bmatrix}, \quad \bar{C} = [C \ 0], \quad \text{and} \quad \bar{E} = \begin{bmatrix} B \\ 0 \end{bmatrix}. \quad (13)$$

We then obtain the overall dynamics of N agents:

$$\dot{\tilde{x}} = [I_N \otimes \bar{A} + G \otimes (\bar{B}\bar{C})]\tilde{x} + (I_N \otimes \bar{E})\rho.$$

Let U be a nonsingular matrix such that $J = U^{-1}GU$ is the Jordan canonical form of G with $J(1, 1) = \lambda_1 = 0$. Define $\eta = [\eta_1; \dots; \eta_N] = (U^{-1} \otimes I_{pn_q+n_c})\tilde{x}$. We then obtain the following dynamical equations for η :

$$\dot{\eta} = [I_N \otimes \bar{A} + J \otimes (\bar{B}\bar{C})]\eta + (U^{-1} \otimes \bar{E})\rho. \quad (14)$$

Lemma 2

Let Assumption 1 hold. If $\bar{A} + \lambda_i \bar{B}\bar{C}$ is Hurwitz stable for all $i \in \{2, \dots, N\}$, then the output synchronization problem for a network of N agents of the form (8) and (9) is solved via N decentralized controllers of the form (11).

Proof

The proof is carried out in two stages. In the first stage, we shall show that the output synchronization problem for a network N agents (8) and (9) via controllers of the form (11) is solved if

$$\lim_{t \rightarrow \infty} \eta_i(t) = 0$$

for all $i \in \{2, \dots, N\}$. We then show that this is guaranteed if $\bar{A} + \lambda_i \bar{B}\bar{C}$ is Hurwitz stable for all $i \in \{2, \dots, N\}$.

Suppose that

$$\lim_{t \rightarrow \infty} \left(\eta(t) - \begin{bmatrix} \eta_1(t) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right) = 0,$$

for some $\eta_1(t) \in \mathbb{C}^{pn_q+n_c}$. Then

$$\begin{aligned} \lim_{t \rightarrow \infty} (\tilde{x}(t) - \mathbf{1} \otimes \eta_1(t)) &= \lim_{t \rightarrow \infty} [(U \otimes I)\eta - (U \otimes I)(U^{-1} \otimes I)(\mathbf{1} \otimes \eta_1(t))] \\ &= (U \otimes I) \lim_{t \rightarrow \infty} [\eta(t) - (U^{-1}\mathbf{1}) \otimes \eta_1(t)] \\ &= (U \otimes I) \lim_{t \rightarrow \infty} \left(\eta(t) - \begin{bmatrix} \eta_1(t) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right) = 0, \end{aligned}$$

where we have used that $U^{-1}\mathbf{1} = [1 \ 0 \ \dots \ 0]^T$, which follows from the fact that $U^{-1}U = I_N$ and that U consists of all the (generalized) right eigenvectors of G , with the first column being $\mathbf{1}$. Hence, the output synchronization is achieved. So far, we have shown that the output

synchronization is achieved if $\lim_{t \rightarrow \infty} \eta_i(t) = 0$ for all $i \in \{2, \dots, N\}$. Next, we shall show that this is ensured if $A + \lambda_i \bar{B}\bar{C}$ is Hurwitz stable for all $i \in \{2, \dots, N\}$.

Define $\bar{\eta} := [\eta_2; \dots; \eta_N]$ and $\omega := [\omega_1; \dots; \omega_N]$, from (14) and (9), we obtain that

$$\begin{bmatrix} \dot{\bar{\eta}} \\ \omega \end{bmatrix} = \begin{bmatrix} I_{N-1} \otimes \bar{A} + \bar{J} \otimes (\bar{B}\bar{C}) & ((\bar{I}U^{-1}) \otimes \bar{E})C_s \\ 0 & A_s \end{bmatrix} \begin{bmatrix} \bar{\eta} \\ \omega \end{bmatrix}, \quad (15)$$

where

$$\bar{I} = \begin{bmatrix} 0 & I_{N-1} \end{bmatrix}, \quad C_s = \text{blkdiag}\{C_{i,s}\}_{i=1}^N, \quad A_s = \text{blkdiag}\{A_{i,s}\}_{i=1}^N, \quad J = \text{blkdiag}\{0; \bar{J}\}.$$

Since $I_{N-1} \otimes \bar{A}$ and $\bar{J} \otimes (\bar{B}\bar{C})$ are block upper triangular, the eigenvalues of $I_{N-1} \otimes \bar{A} + \bar{J} \otimes (\bar{B}\bar{C})$ are the union of eigenvalues $\bar{A} + \lambda_i \bar{B}\bar{C}$ for $i \in \{2, \dots, N\}$, which are in the open left-half complex plane by the assumption. Together with the fact that A_s is Hurwitz stable, it is clear that the system (15) is asymptotically stable, that is, $\lim_{t \rightarrow \infty} \bar{\eta}(t) = 0$ for any initial conditions $\bar{\eta}(0)$ and $\omega(0)$. $\square$

Remark 5

In view of Lemma 2, the dynamics of $\eta_1(t)$ is governed by

$$\dot{\eta}_1(t) = \bar{A}\eta_1(t) + (v' \otimes \bar{E})\rho(t), \quad \eta_1(0) = (v' \otimes I_{pn_q+n_c})\tilde{x}(0),$$

where v' is the first row of the matrix U^{-1} , i.e., the left eigenvector corresponding to the eigenvalue of G at zero. Since $\rho(t)$ is exponentially decaying, from [27, Lemma B.1] and Lemma 2, we see that for each $i \in \{1, \dots, N\}$,

$$\lim_{t \rightarrow \infty} (\tilde{x}_i(t) - e^{\bar{A}t}\tilde{\eta}_1) = \lim_{t \rightarrow \infty} [(\tilde{x}_i(t) - \eta_1(t)) + (\eta_1(t) - e^{\bar{A}t}\tilde{\eta}_1)] = 0, \quad (16)$$

for some $\tilde{\eta}_1 \in \mathbb{R}^{pn_q+n_c}$. $\square$

From Lemma 2, we see that the output synchronization problem for a network of agents (8) and (9) is achieved if $\bar{A} + \lambda_i \bar{B}\bar{C}$ is Hurwitz stable, for all $i \in \{2, \dots, N\}$, which is a simultaneous stabilization problem. More specifically, we need to design the parameters A_k , B_k , and C_k in (11), such that the following compensator

$$\begin{cases} \dot{\chi} = A_k\chi + B_kz, \\ u = C_k\chi, \end{cases} \quad (17)$$

simultaneously stabilizes all the $N - 1$ systems given by

$$\begin{cases} \dot{x} = Ax + Bu, \\ z = \lambda_i Cx, \quad i \in \{2, \dots, N\}. \end{cases} \quad (18)$$

Due to the linearity, it is easy to see that the compensator (17) simultaneously stabilizes (18) if it simultaneously stabilizes all the $N - 1$ systems given by

$$\begin{cases} \dot{x} = Ax + \lambda_i Bu, \\ z = Cx, \quad i \in \{2, \dots, N\}. \end{cases} \quad (19)$$

Lemma 3

The output synchronization for a heterogeneous network of N agents (5) as defined in Problem 1 is solvable if (17) simultaneously stabilizes all the $N - 1$ systems (19).

Proof

If (17) simultaneously stabilizes all the $N - 1$ systems (19), then the composition of (7) and (11), which is of the form (6), solves Problem 1. $\square$

Lemma 3 converts the output synchronization problem for a heterogeneous network of N agents (5) as defined in Problem 1 to a simultaneous stabilization problem.

3.3. Simultaneous stabilization via a low-gain approach

In this section, we design the parameters A_k, B_k and C_k of the compensator (11), such that the compensator (17) simultaneously stabilizes all the $N - 1$ systems (19).

It is clear that we can choose the matrix H in (10) such that the matrix A has all the eigenvalues on the imaginary axis.

Given a $\beta > 0$, such that $\text{Re}(\lambda_2(G)) \geq \beta$ for any $\mathcal{G} \in \Gamma_{\beta, \gamma}$, let $P(\varepsilon) = P'(\varepsilon) > 0$ be the unique solution of the continuous-time algebraic Riccati equation

$$A'P(\varepsilon) + P(\varepsilon)A - \beta P(\varepsilon)BB'P(\varepsilon) + \varepsilon I_{pn_q} = 0. \quad (20)$$

We then design the controller of the form (11) as:

$$\begin{cases} \dot{\chi}_i = A_k \chi_i + B_k \zeta_i := (A + KC)\chi_i - K\zeta_i, \\ v_i = C_k \chi_i := B'P(\varepsilon)\chi_i, \quad i \in \{1, \dots, N\}, \end{cases} \quad (21)$$

where the matrix K is such that $A + KC$ is Hurwitz stable, and $\varepsilon > 0$ is a low-gain parameter. Note that (21) is of CSS type observer, see [3].

Following the proof of [23, Theorem 4], with just a little bit modification, we see that there exists an ε^* , which depends on γ , such that for $\varepsilon \in (0, \varepsilon^*]$, the compensator (17) with the parameters A_k, B_k and C_k given by (21) simultaneously stabilizes all the $N - 1$ systems (19).

Remark 6

Note that the matrix A_k in the controller (21) is Hurwitz stable and the matrix $\bar{A}$ given by (13) is block upper triangular. It is then follows from the result of [27, Lemma B.1] and Remark 5 that the output synchronization trajectory is given by

$$\lim_{t \rightarrow \infty} (y_i(t) - Ce^{At}d) = 0, \quad \forall i \in \{1, \dots, N\}.$$

for some $d \in \mathbb{R}^{pn_q}$.

4. APPLICATION TO OUTPUT FORMATION

In this section, we consider the output formation problem to be formally defined shortly. We shall show that the output formation problem can be solved by slightly modifying the design procedure for solving the output synchronization problem as defined in Problem 1.

Definition 3

An output formation is a family of vectors $\{h_1, \dots, h_N\}$, $h_i \in \mathbb{R}^p$, $i \in \{1, \dots, N\}$. The heterogeneous network of N agents (5) is said to achieve the output formation if

$$\lim_{t \rightarrow \infty} [(y_i(t) - h_i) - (y_j(t) - h_j)] = 0, \quad \forall i, j \in \{1, \dots, N\}. \quad (22)$$

For this problem, we assume that the network infrastructure provides each agent with the following information

$$\hat{\zeta}_i = \sum_{j=1}^N a_{ij}[(y_i - h_i) - (y_j - h_j)] = \sum_{j=1}^N g_{ij}(y_j - h_j), \quad (23)$$

With the local information z_i given by (2) and the information $\hat{\zeta}_i$ given by (23) provided by the network, the agent i , where $i \in \{1, \dots, N\}$, has the following dynamical equations:

$$\begin{cases} \dot{x}_i = A_i x_i + B_i u_i, \\ y_i = C_i x_i, \\ z_i = C_i^m x_i, \\ \hat{\zeta}_i = \sum_{j=1}^N g_{ij}(y_j - h_j). \end{cases} \quad (24)$$

Let us formally formulate the output formation problem for a heterogeneous network.

Problem 2 (Output formation)

Consider a heterogeneous network of N agents (24). For any given $\gamma \geq \beta > 0$ and the resulting set $\Gamma_{\beta,\gamma}$, and an arbitrarily given family of vectors $\{h_1, \dots, h_N\}$, where $h_i \in \mathbb{R}^p$ for $i \in \{1, \dots, N\}$, the output formation problem with a set of communication topologies $\Gamma_{\beta,\gamma}$ is to find, if possible, a linear dynamical controller

$$\begin{cases} \dot{x}_{i,c} = A_{i,c}x_{i,c} + B_{i,c}\hat{\zeta}_i + E_{i,c}z_i, \\ u_i = C_{i,c}x_{i,c} + D_{i,c}\hat{\zeta}_i + M_{i,c}z_i, \end{cases} \quad (25)$$

such that the output formation as defined in Definition 3 is achieved for any network communication topology $\mathcal{G} \in \Gamma_{\beta,\gamma}$.

Theorem 2

Consider a heterogeneous network of N agents (24). Let Assumptions 1 and 2 hold. Then the output formation problem with a set of communication topologies $\Gamma_{\beta,\gamma}$ for any $\gamma \geq \beta > 0$, and any formation vectors $\{h_1, \dots, h_N\}$, where $h_i \in \mathbb{R}^p$ for $i \in \{1, \dots, N\}$, as defined in Problem 2, is solvable via N decentralized controllers of the form (25).

The proof of Theorem 2 is very similar to the proof of Theorem 1 by explicit construction of a formation controller of the form (25). We first design a local pre-compensator of the form (7) for each agent such that the resulting systems are almost identical, that is, all the resulting systems are characterized by the same triple (C, A, B) for which the output formation is always achievable. The following lemma shows the existence of such a triple (C, A, B) .

Lemma 4

For an arbitrarily given family of vectors $\{h_1, \dots, h_N\}$, $h_i \in \mathbb{R}^p$, $i = 1, \dots, N$ and an integer $n_q > 0$, there exist a triple (C, A, B) and another family of vectors $\{\bar{h}_1, \dots, \bar{h}_N\}$ of appropriate dimensions, such that

1. $\text{rank}(C) = p$,
2. (C, A, B) is invertible, of uniform rank n_q , and has no invariant zero,
3. A has all its eigenvalues in the closed left-half complex plane,
4. $A\bar{h}_i = 0$,
5. $C\bar{h}_i = h_i$.

Proof

Since we have freedom to choose the triple (C, A, B) in Lemma 1, let us choose the triple (C, A, B) as follows:

$$A = A_0 + B_0H, \quad B = B_0, \quad C = C_0,$$

where A_0, B_0, C_0 are given in (10), $H = \begin{bmatrix} 0 & H_0 \end{bmatrix}$, and the matrix H_0 is such that the matrix $\bar{A}_0 + \bar{B}_0H_0$, where

$$\bar{A}_0 := \begin{bmatrix} 0 & I_{p(n_q-2)} \\ 0 & 0 \end{bmatrix}, \quad \bar{B}_0 := \begin{bmatrix} 0 \\ I_p \end{bmatrix},$$

has all the eigenvalues in the closed left-half complex plane. Such an H_0 exists due to the fact that $(\bar{A}_0, \bar{B}_0)$ is controllable. It is then easy to see that the matrix $A_0 + B_0H$ has $p(n_q - 1)$ eigenvalues, which are the eigenvalues of $\bar{A}_0 + \bar{B}_0H_0$, and the remaining p eigenvalues are simple eigenvalues at zero. Therefore, the third condition is satisfied.

We then define a family of vectors $\{\bar{h}_1, \dots, \bar{h}_N\}$ as follows:

$$\bar{h}_i = \begin{bmatrix} h_i \\ 0 \end{bmatrix}, \quad i = 1, \dots, N.$$

It is then easy to see that

$$C\bar{h}_i = C_0 \begin{bmatrix} h_i \\ 0 \end{bmatrix} = \begin{bmatrix} I_p & 0 \end{bmatrix} \begin{bmatrix} h_i \\ 0 \end{bmatrix} = h_i,$$

and

$$A\bar{h}_i = (A_0 + B_0H) \begin{bmatrix} h_i \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & I_{p(n_q-1)} \\ 0 & H_0 \end{bmatrix} \begin{bmatrix} h_i \\ 0 \end{bmatrix} = 0.$$

□

Proof of Theorem 2

For any triple (C, A, B) which satisfies the condition of Lemma 4, from Lemma 1, it is clear that we can design a pre-compensator of the form (7) for each agent, such that the interconnection of (5) and (7) can be written in the following form:

$$\begin{cases} \dot{\bar{x}}_i = A\bar{x}_i + B(v_i + \rho_i), \\ y_i = C\bar{x}_i, \\ \hat{\zeta}_i = \sum_{j=1}^N g_{ij}(y_j - h_j), \end{cases} \quad (26)$$

where ρ_i is given by (9).

Define $\bar{x}_{i,s} = \bar{x}_i - \bar{h}_i$, since $A\bar{h}_i = 0$ and $C\bar{h}_i = h_i$ for $i = 1, \dots, N$, (26) can be rewritten in term of $\bar{x}_{i,s}$ as:

$$\begin{cases} \dot{\bar{x}}_{i,s} = A\bar{x}_{i,s} + B(v_i + \rho_i), \\ y_i = C\bar{x}_{i,s} + h_i, \\ \hat{\zeta}_i = \sum_{j=1}^N g_{ij}(y_j - h_j). \end{cases} \quad (27)$$

Following the design procedure given in Section 3.3, we then design the following decentralized controller for each agent

$$\begin{cases} \dot{\chi}_i = (A + KC)\chi_i - K\hat{\zeta}_i, \\ v_i = B'P(\varepsilon)\chi_i, \end{cases} \quad (28)$$

where the matrix K is such that $A + KC$ is Hurwitz stable, $\varepsilon > 0$ is a low-gain parameter, and $P(\varepsilon) = P'(\varepsilon) > 0$ is the unique solution of the algebraic Riccati equation (20).

It then follows from the analysis in Section 3.3 that there exists an ε^* , which depends on γ , such that for all $\varepsilon \in (0, \varepsilon^*]$, the controller (28) solve the output synchronization for a network of N the models (27). Hence, $\lim_{t \rightarrow \infty} [(y_i(t) - h_i) - (y_j(t) - h_j)] = \lim_{t \rightarrow \infty} (C\bar{x}_{i,s}(t) - C\bar{x}_{j,s}(t)) = 0$ for all $i, j \in \{1, \dots, N\}$. □

5. REGULATION OF OUTPUT SYNCHRONIZATION

Note that the output synchronization problem does not impose any conditions on asymptotic behavior of the outputs of the agent models as long as they are asymptotic identical. In this section, we consider the related problem of regulating the output towards a desired reference trajectory $y_r(t)$, generated by an autonomous exosystem

$$\begin{cases} \dot{x}_r = A_r x_r, & x_r(0) = x_{r0}, \\ y_r = C_r x_r, \end{cases} \quad (29)$$

where $x_r \in \mathbb{R}^r$ and $y_r \in \mathbb{R}^p$.

We make the following assumption about the exosystem (29):

Assumption 3

For the exosystem (29),

1. (C_r, A_r) is observable,
2. All the eigenvalues of A_r are on the imaginary axis.

Definition 4

A heterogeneous network of N agents is said to achieve the regulation of output synchronization if

$$\lim_{t \rightarrow \infty} (y_i(t) - y_r(t)) = 0, \quad \forall i = \{1, \dots, N\}.$$

For solving the regulation of output synchronization problem, we consider a subset Γ_s of Γ , where Γ is the set of all the network topologies, each of which contains a directed spanning tree. We assume that all the topologies in the set Γ_s have a common root. Without loss of generality, we assume that the common root is node (agent) 1. This (root) agent 1 measures its own output relative to the output of the exosystem, that is, agent 1 has access to a quantity $\psi_1 = d(y_1 - y_r)$, where $d > 0$, while $\psi_i = 0$ for all $i \in \{2, \dots, N\}$.

With the local information z_i given by (2), the information ζ_i given by (3) provided by the network, and information ψ_i , the agent i for $i \in \{1, \dots, N\}$ has the following dynamical equations:

$$\begin{cases} \dot{x}_i = A_i x_i + B_i u_i, \\ y_i = C_i x_i, \\ z_i = C_i^m x_i, \\ \bar{\zeta}_i = \sum_{j=1}^N g_{ij} y_j + \psi_i. \end{cases} \quad (30)$$

Let us now formally formulate the regulation of output synchronization problem.

Problem 3 (Regulation of Output Synchronization)

Consider a heterogeneous network of N agents (30) and the autonomous exosystem (29). For any given set $\Gamma_s \subset \Gamma$, the regulation of output synchronization problem is to find, if possible, a linear dynamical controller

$$\begin{cases} \dot{x}_{i,c} = A_{i,c} x_{i,c} + B_{i,c} \bar{\zeta}_i + E_{i,c} z_i, \\ u_i = C_{i,c} x_{i,c} + D_{i,c} \bar{\zeta}_i + M_{i,c} z_i, \end{cases} \quad (31)$$

for each agent $i \in \{1, \dots, N\}$, such that regulation of output synchronization is achieved for any network communication topology represented by the digraph $\mathcal{G} \in \Gamma_s$.

We present some preliminary work which are needed for presenting the result for the regulation of output synchronization problem as defined in Problem 3. Let $\bar{\mathcal{G}}$ denote an expanded network constructed from $\mathcal{G} \in \Gamma_s$ by adding the exosystem as node 0 and the edge from exosystem to agent 1 with weight d . It is then easy to see that the Laplacian matrix of the network $\bar{\mathcal{G}}$ is given by

$$\bar{G} = [\bar{g}_{ij}] = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ -d & g_{11} + d & g_{12} & \cdots & g_{1N} \\ 0 & g_{21} & g_{22} & \cdots & g_{2N} \\ \vdots & \vdots & \cdots & \ddots & \vdots \\ 0 & g_{N1} & g_{N2} & \cdots & g_{NN} \end{bmatrix}. \quad (32)$$

In view of (32), $\bar{\zeta}_i$ in (30) can be rewritten as

$$\bar{\zeta}_i = \sum_{j=1}^N g_{ij} y_j + \psi_i = \sum_{j=0}^N \bar{g}_{ij} y_j. \quad (33)$$

Also note that the expanded network also contains a directed spanning tree rooted at the node 0. It is then easy to see from [16, Lemma 3.3] that all the eigenvalues of $\bar{G}$ are in the closed right-half complex plane. Let $\bar{\lambda}_1, \dots, \bar{\lambda}_{N+1}$ denote the eigenvalues of $\bar{G}$, such that $\bar{\lambda}_1 = 0$ and $0 < \text{Re}(\bar{\lambda}_2) \leq \dots \leq \text{Re}(\bar{\lambda}_{N+1})$.

Assumption 4

There exist $\bar{\gamma} \geq \bar{\beta} > 0$, such that for each expanded network, the corresponding Laplacian matrix has following properties:

1. $\text{Re}(\bar{\lambda}_2) \geq \bar{\beta} > 0$;
2. $\max_{i=2, \dots, N+1} |\bar{\lambda}_i| \leq \bar{\gamma}$.

We are now ready to present our result for the regulation of output synchronization.

Theorem 3

Consider a heterogeneous network of N agents (30) and the autonomous exosystem (29). Let Assumptions 1, 2, 3 and 4 hold. Then the regulation of output synchronization problem as defined in Problem 3 is solvable via N decentralized controllers of the form (31).

Proof

For an exosystem given by (29), it is shown in Appendix C that there exist another exosystem given by

$$\begin{cases} \dot{\tilde{x}}_r = \tilde{A}_r \tilde{x}_r, & \tilde{x}_r(0) = \tilde{x}_{r0}, \\ y_r = \tilde{C}_r \tilde{x}_r, \end{cases} \quad (34)$$

such that for all $x_{r0} \in R^r$ there exists $\tilde{x}_{r0} \in R^{\tilde{r}}$ for which (34) generates exact the same output y_r as the original exosystem (29). Furthermore, we can find a matrix $\tilde{B}_r$ such that the triple $(\tilde{C}_r, \tilde{A}_r, \tilde{B}_r)$ is invertible, of uniform rank n_q , and has no invariant zero, where n_q is an integer greater than or equal to maximal order of infinite zeros of (C_i, A_i, B_i) , $i \in \{1, \dots, N\}$ and all the observability index (see [2] for the definition) of (C_r, A_r) . Note that as seen from Appendix C, the eigenvalues of $\tilde{A}_r$ consists of all the eigenvalues of A_r and additional zero eigenvalues, which are degenerate.

The new exosystem can be rewritten as:

$$\begin{cases} \dot{\tilde{x}}_r = \tilde{A}_r \tilde{x}_r + \tilde{B}_r(v_r + \rho_r), & \tilde{x}_r(0) = \tilde{x}_{r0}, \\ y_r = \tilde{C}_r \tilde{x}_r, \end{cases} \quad (35)$$

where $v_r(t) = 0$ and $\rho_r(t) = 0$.

Following from the constructive proof of Lemma 1, we then design a pre-compensator (7) for each agent $i \in \{1, \dots, N\}$ such that the interconnection of (5) and (7) are almost identical to the exosystem system (35), that is, for each agent $i \in \{1, \dots, N\}$,

$$\begin{cases} \dot{\tilde{x}}_i = \tilde{A}_r \tilde{x}_i + \tilde{B}_r(v_i + \rho_i), \\ y_i = \tilde{C}_r \tilde{x}_i, \\ \bar{\zeta}_i = \sum_{j=0}^N \bar{g}_{ij} y_j, \end{cases} \quad (36)$$

where ρ_i is given by (9).

It is then easy to see that regulation of output synchronization for a heterogeneous network of N agents is converted to the output synchronization problem for an expanded network of $N + 1$ agents by adding the exosystem system as agent 0 and the edge from agent 0 to agent 1 with weight d . More specifically, define $\tilde{x}_0 := \tilde{x}_r$, $y_0 := \tilde{y}_r$, $v_0 := v_r$, and $\rho_0 := \rho_r$, the agent i , where $i \in \{0, 1, \dots, N\}$ has the following dynamics:

$$\begin{cases} \dot{\tilde{x}}_i = \tilde{A}_r \tilde{x}_i + \tilde{B}_r(v_i + \rho_i), \\ y_i = \tilde{C}_r \tilde{x}_i, \\ \bar{\zeta}_i = \sum_{j=0}^N \bar{g}_{ij} y_j. \end{cases} \quad (37)$$

Following the design procedure given in Section 3.3, we design the following controller for each agent $i \in \{0, 1, \dots, N\}$

$$\begin{cases} \dot{\chi}_i = (\tilde{A}_r + K\tilde{C}_r)\chi_i - K\bar{\zeta}_i, \\ v_i = \tilde{B}_r' P(\varepsilon)\chi_i, \end{cases} \quad (38)$$

where the matrix K is such that $A + KC$ is Hurwitz stable, $\varepsilon > 0$ is a low-gain parameter, and $P(\varepsilon) = P'(\varepsilon) > 0$ is the unique solution of the following continuous-time algebraic Riccati equation

$$\tilde{A}_r' P(\varepsilon) + P(\varepsilon) \tilde{A}_r - \beta P(\varepsilon) \tilde{B}_r \tilde{B}_r' P(\varepsilon) + \varepsilon I_{pn_q} = 0. \quad (39)$$

Note that in the controller (38), we choose $\chi_0(0) = 0$ for agent 0. It is then clear that $v_0(t) = 0$ as desired since $\bar{\zeta}_0(t) = 0$.

It then follows from the analysis in Section 3.3 that there exists an ε^* , which depends on $\tilde{\gamma}$, such that for all $\varepsilon \in (0, \varepsilon^*]$, the controller (38), solves the output synchronization for a set of the expanded network topologies. Hence, $\lim_{t \rightarrow \infty} (y_i(t) - y_r(t)) = 0$ for all $i \in \{1, \dots, N\}$. $\square$

6. ILLUSTRATIVE EXAMPLE

6.1. Output synchronization

We illustrate our design procedure on a network of four agents. The agents dynamics are of form (1) with

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C_1 = [1 \ 0 \ 0 \ 0], \quad C_1^m = [1 \ 1 \ 0 \ 0],$$

$$A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C_2 = [1 \ 0 \ 0], \quad C_2^m = [1 \ 1 \ 0],$$

$$A_i = \begin{bmatrix} -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 1 & 1 \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad C_i = [0 \ 0 \ 0 \ 1 \ 0], \quad C_i^m = [1 \ 1 \ 0 \ 0 \ 0],$$

for $i = 3, 4$.

Given $\beta = 2.8$ and $\gamma = 4.1$, we have the resulting set $\Gamma_{2.8,4.1}$. Two network topologies in this set are given by Figure 1.

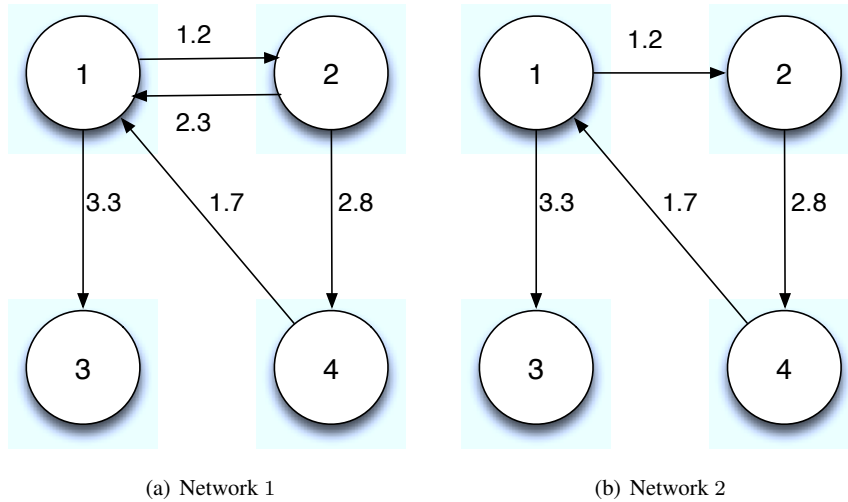


Figure 1. Network Topologies

Note that $\bar{n}_d = 3$, which is the degree of the infinite zeros of (C_2, A_2, B_2) . We then choose $n_q = 3$, and matrices A, B, C as below

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = [1 \ 0 \ 0].$$

It is easy to see that the above matrices A, B, C satisfy the conditions of Lemma 1. Let us choose $\varepsilon = 0.01$ and

$$K = \begin{bmatrix} -6 \\ -10 \\ 0 \end{bmatrix}$$

and design the dynamic low-gain controller as follows:

$$\begin{cases} \dot{\chi}_i(t) = \begin{bmatrix} -6 & 1 & 0 \\ -10 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \chi_i(t) - \begin{bmatrix} -6 \\ -10 \\ 0 \end{bmatrix} \zeta_i(t) \\ v^i(t) = -[0.0598 \ 0.0183 \ 0.1423] \chi_i(t) \end{cases} \quad (40)$$

Figure 2 and Figure 3 show that the output synchronization is achieved for Network 1 and Network 2, respectively.

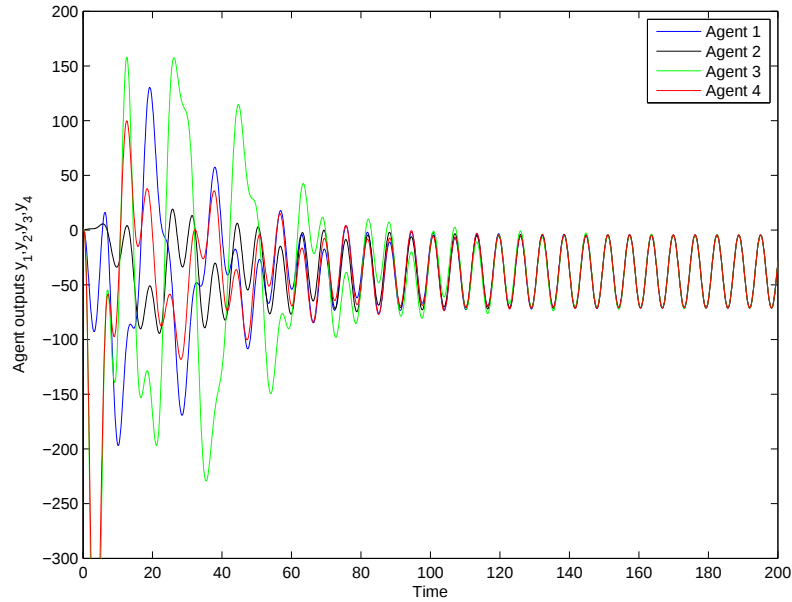


Figure 2. Outputs for Network 1

6.2. Output formation

Consider the same two networks as in Section 6.1, our goal is to achieve output formation. We choose $h_1 = 10$, $h_2 = 20$, $h_3 = 30$, and $h_4 = 40$. Figure 4 and Figure 5 show that the output formation is achieved[‡] for Network 1 and Network 2, respectively.

6.3. Regulation of output synchronization

Consider the same network as in Section 6.1, however, our goal now is to ensure that each agent's output follows the output y_r of the following exosystem

$$\begin{cases} \dot{x}_r = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_r, \\ y_r = \begin{bmatrix} 1 & 0 \end{bmatrix} x_r, \end{cases}$$

with $x_r(0) = [1; 1]$.

[‡]Note that x_{d1} , x_{d2} , and x_{d3} is the coordinate where all the agents are almost identical

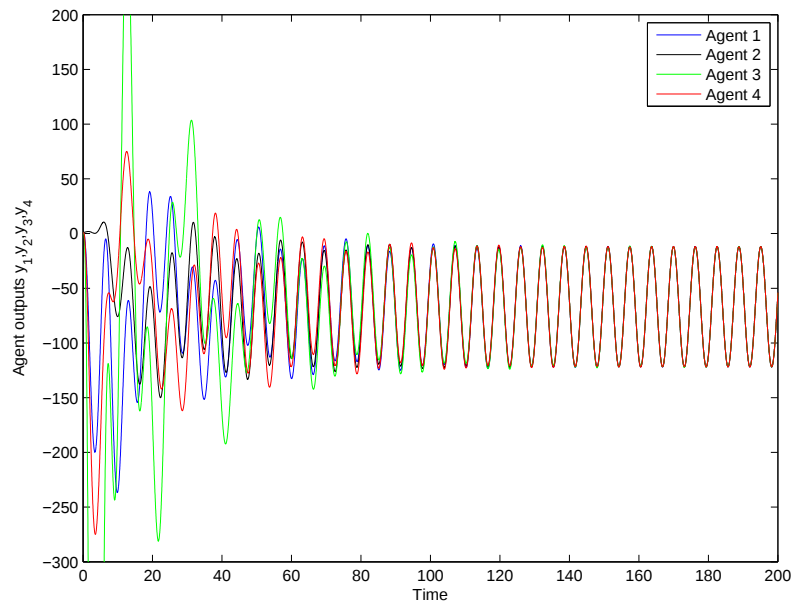


Figure 3. Outputs for Network 2

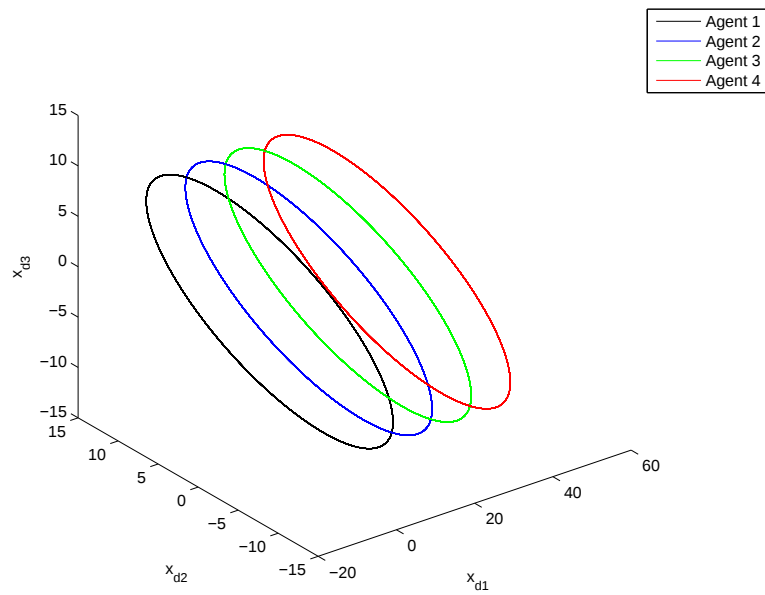


Figure 4. Formation for Network 1

We first expand the system to the following form

$$\begin{cases} \dot{\tilde{x}}_r = \tilde{A}_r \tilde{x}_r := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \tilde{x}_r, \\ y_r = \tilde{C}_r \tilde{x}_r := \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \tilde{x}_r, \end{cases}$$

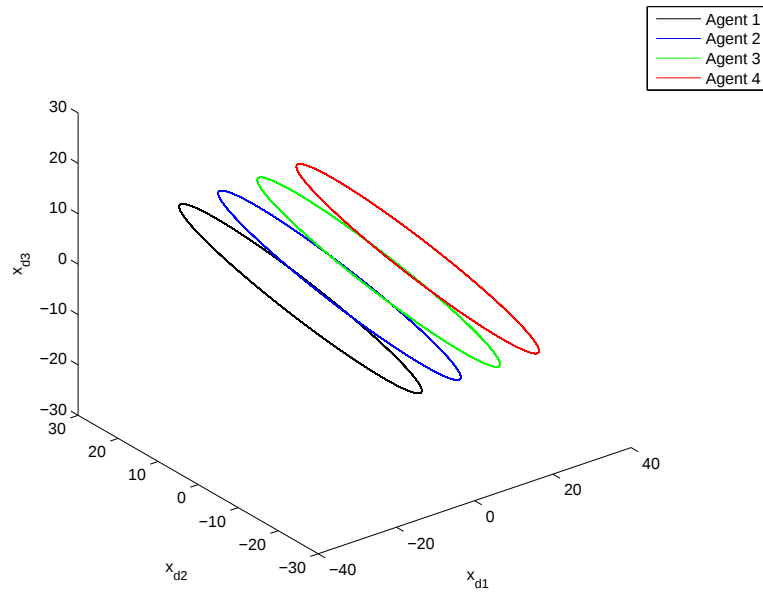


Figure 5. Formation for Network 2

with $\tilde{x}_r(0) = [1; 1; 0]$.

Let us now choose $\tilde{B}_r = [0 \ 0 \ 1]'$. We then follow the same design procedure to design precompensator to make all the agents almost identical with different exponentially decaying signals. We then add a link with weight 10 from the exosystem to the root agent 1 for Networks 1 and 2 whose topologies are given by Figure 1. The resulting network topologies are shown in Figure 6.

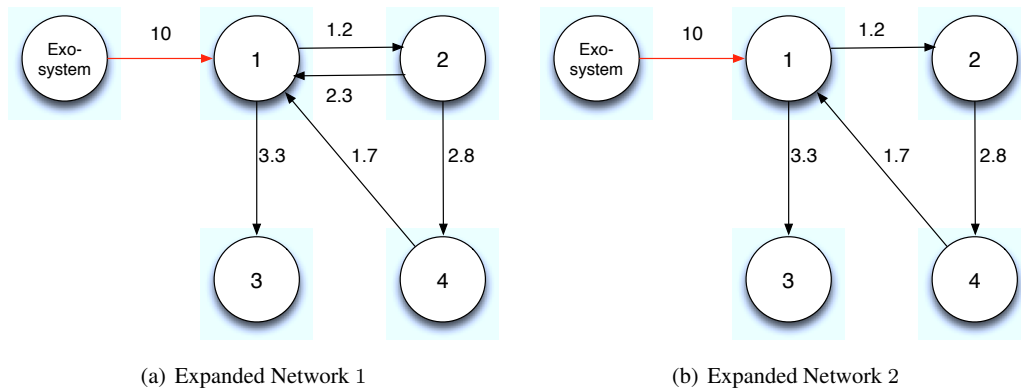


Figure 6. Expanded Network Topologies

Choose $\bar{\beta} = 0.77$, $\bar{\gamma} = 14.18$, and $\varepsilon = 10^{-8}$. Figures 7 and 8 show the regulation of output synchronization is achieved for Network 1 and Network 2, respectively.

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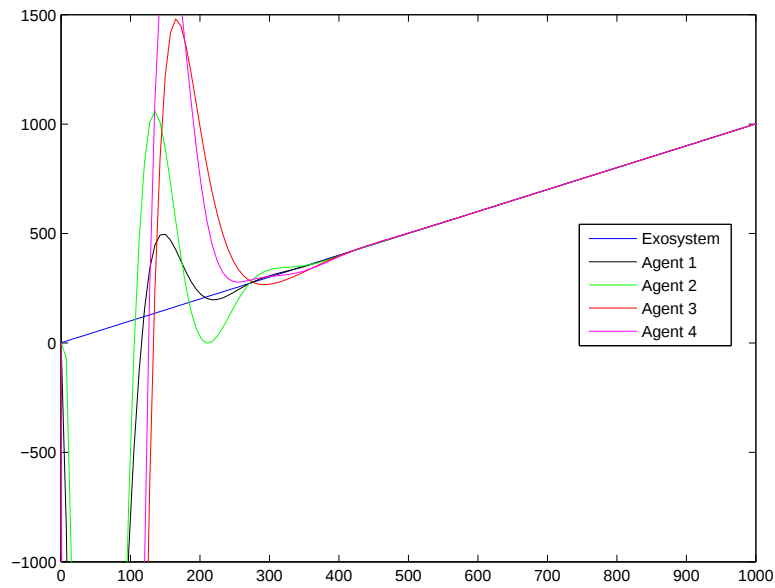


Figure 7. Outputs for expanded Network 1

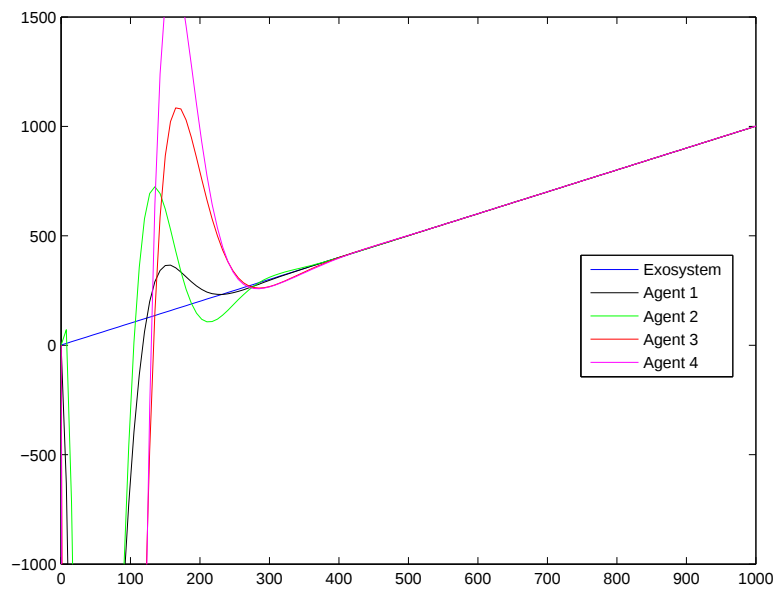


Figure 8. Outputs for expanded Network 2

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A. PRELIMINARY

In order to better understand our design methodology, the readers need to get familiar with special coordinate basis (SCB)[22], how to square down the right invertible system, and how to make the invertible system uniform rank. Therefore, we will briefly review these materials.

A.1. Review of SCB

Consider a strictly proper linear system given by

$$\begin{cases} \dot{x} = Ax + Bu, \\ y = Cx, \end{cases} \quad (41)$$

with B injective where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^p$. There exist nonsingular transformations Γ_s , Γ_o , and Γ_i , such that

$$\begin{aligned} x &= \Gamma_s \tilde{x}, \quad y = \Gamma_o \tilde{y}, \quad u = \Gamma_i \tilde{u}, \\ \tilde{x} &= \begin{bmatrix} x_a \\ x_b \\ x_c \\ x_d \end{bmatrix}, \quad \tilde{y} = \begin{bmatrix} y_d \\ y_b \end{bmatrix}, \quad \tilde{u} = \begin{bmatrix} u_d \\ u_c \end{bmatrix}, \\ x_d &= \begin{bmatrix} x_{d,1} \\ x_{d,2} \\ \vdots \\ x_{d,m_d} \end{bmatrix}, \quad y_d = \begin{bmatrix} y_{d,1} \\ y_{d,2} \\ \vdots \\ y_{d,m_d} \end{bmatrix}, \quad u_d = \begin{bmatrix} u_{d,1} \\ u_{d,2} \\ \vdots \\ u_{d,m_d} \end{bmatrix}, \end{aligned}$$

and that in the new coordinate, (41) can be rewritten as

$$\begin{cases} \dot{x}_a = A_{aa}x_a + L_{ab}y_b + L_{ad}y_d, \\ \dot{x}_b = A_{bb}x_b + L_{bd}y_d, \\ \dot{x}_c = A_{cc}x_c + B_c(u_c + E_{ca}x_a) + L_{cb}y_b + L_{cd}y_d, \\ \dot{x}_{d,j} = A_{d,j}x_{d,j} + B_{d,j}(u_{d,j} + E_{d,j,a}x_a + E_{d,j,b}x_b + E_{d,j,c}x_c) + L_{d,j}y_d, \\ y_{d,j} = C_{d,j}x_{d,j}, \quad j = 1, \dots, m_d \\ y_b = C_b x_b. \end{cases} \quad (42)$$

Here the states x_a , x_b , x_c , and x_d are respectively of dimensions n_a , n_b , n_c , and $n_d = \sum_{j=1}^{m_d} n_{d,j}$, while the state $x_{d,j}$ is of dimension $n_{d,j}$ for each $j = 1, \dots, m_d$. The inputs u_d and u_c are respectively of dimensions m_d and $m_c = m - m_d$, while the outputs y_d and y_c are respectively of dimensions m_d and $p - m_d$. The matrices $A_{d,j}$, $B_{d,j}$ and $C_{d,j}$ have the form

$$A_{d,j} = \begin{bmatrix} 0 & I_{n_{d,j}-1} \\ 0 & 0 \end{bmatrix}, \quad B_{d,j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_{d,j} = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Some important properties of SCB are summarized as follows:

1. The invariant zeros of the system (41) are the eigenvalues of A_{aa} .
2. (A_{cc}, B_c) is controllable, and (C_b, A_{bb}) is observable.
3. If the system (41) is right-invertible, then x_b , and hence y_b are nonexistence, and $\Gamma_o = I$.
4. If the system (41) is left-invertible, then x_c , and hence u_c are nonexistence, and $\Gamma_i = I$.
5. The system (41) has m_d zeros at the infinity with the order $n_{d,j}$, $j = 1, \dots, m_d$.

A.2. Squaring-down for a right-invertible system

Let us now recall the following result from [21]:

Lemma 5

Assume that for the system (41), (A, B) is stabilizable, (C, A) is detectable, and (C, A, B) is right-invertible, then there exists a precompensator of the form

$$\begin{cases} \dot{\chi}^1 = A^1 \chi^1 + B^1 u^1, \\ u = C^1 \chi^1 + D^1 u^1, \end{cases} \quad (43)$$

such that the resulting system of (41) and (43) is invertible.

The proof was given in [21] by explicit construction of such a precompensator. To be self contained, we briefly review such a design procedure.

If the system (41) is right-invertible, then x_b , and hence y_b are nonexistence. Therefore, with nonsingular transformations Γ_s and Γ_i , the system (41) can be transformed into the following SCB form:

$$\begin{cases} \dot{x}_a = A_{aa}x_a + L_{ad}y_d, \\ \dot{x}_c = A_{cc}x_c + B_c(u_c + E_{ca}x_a) + L_{cd}y_d, \\ \dot{x}_{d,j} = A_{d,j}x_{d,j} + B_{d,j}(u_{d,j} + E_{d,j,a}x_a + E_{d,j,c}x_c) + L_{d,j}y_d, \\ y_{d,j} = C_{d,j}x_{d,j}, \quad j = 1, \dots, m_d \end{cases} \quad (44)$$

Consider the following precompensator for the system (44)

$$\begin{cases} \dot{\chi}^1 = N' \chi^1 + G' u^1, \\ \tilde{u} = [WE'_c, M]' \chi^1 + [I, J]' u^1, \end{cases} \quad (45)$$

where $E_c = [E'_{d,1,c}, E'_{d,2,c}, \dots, E'_{d,m_d,c}]' \in \mathbb{R}^{n_c \times m_d}$, $\chi^1 \in \mathbb{R}^{n_c - m_c}$,

$$WA'_{cc} = NW + MB'_c, \quad \text{rank} \begin{bmatrix} W \\ B'_c \end{bmatrix} = n_c, \quad [G, J]' = \begin{bmatrix} W \\ B'_c \end{bmatrix}^{-1} K,$$

and N and $A_{cc} - KE_c$ are Hurwitz stable. Such a matrix K exists since (A_{cc}, E_c) is detectable, which follows from the fact that (C, A) is detectable.

In [21], it is shown that the resulting system of (44) and (45) is invertible, and has the same infinite zero structure as the system (44). Moreover, the design procedure introduced additional invariant zeros, which are eigenvalues of N and $A_{cc} - KE_c$, and hence can be assigned to the open left-half plane.

It is then easy to see that the precompensator of the form (43) for the system (41) is given by

$$\begin{cases} \dot{\chi}^1 = N' \chi^1 + G' u^1, \\ u = \Gamma_i [WE'_c, M]' \chi^1 + \Gamma_i [I, J]' u^1. \end{cases} \quad (46)$$

A.3. Rank-equalization for a invertible system

Let us now recall the following result from [20]:

Lemma 6

Assume that the system (41) is invertible, then there exists a precompensator of the form

$$\begin{cases} \dot{\chi}^2 = A^2 \chi^2 + B^2 u^2, \\ u = C^2 \chi^2 + D^2 u^2, \end{cases} \quad (47)$$

such that the resulting system of (41) and (47) is uniform rank.

The proof is given in [20]. The idea is to add an appropriate number of integrators to each scalar input $u_{d,j}$ for $j = 1, \dots, m_d$. Let us briefly review such a design.

Since the system (41) is invertible, with a nonsingular transformation Γ_s , the system (41) can be transformed into the following SCB form:

$$\begin{cases} \dot{x}_a = A_{aa}x_a + L_{ad}y_d, \\ \dot{x}_{d,j} = A_{d,j}x_{d,j} + B_{d,j}(u_{d,j} + E_{d,j,a}x_a) + L_{d,j}y_d, \\ y_{d,j} = C_{d,j}x_{d,j}, \quad j = 1, \dots, m_d. \end{cases} \quad (48)$$

Let $\bar{r} \geq \max_{j=1, \dots, m_d} n_{d,j}$. We then design the following pre-compensator for the system (48)

$$\begin{cases} \chi_j^2 = A_j^2 x_j^2 + B_j^2 u_j^2, \\ u_{d,j} = C_j^2 x_j^2 + D_j^2 u_j^2. \end{cases} \quad (49)$$

Here for the chain j where $n_{d,j} < \bar{r}$,

$$A_j^2 = \begin{bmatrix} 0 & I_{\bar{r}-n_{d,j}-1} \\ 0 & 0 \end{bmatrix}, \quad B_j^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_j^2 = [1 \quad 0], \quad D_j^2 = 0,$$

while for the chain j where $n_{d,j} = \bar{r}$, χ_j^2 , and hence A_j^2 , B_j^2 and C_j^2 are nonexistence, while $D_j^2 = 1$, that is $u_{d,j} = u_j^2$.

It is then easy to see that the precompensator of the form (47) for the system (41) is given by

$$\begin{cases} \dot{\chi}^2 = A^2 \chi^2 + B^2 u^2, \\ u = C^2 \chi^2 + D^2 u^2, \end{cases}$$

where $\chi^2 = [\chi_1^2; \dots; \chi_{m_d}^2]$, $u^2 = [u_1^2; \dots; u_{m_d}^2]$,

$$A^2 = \text{blkdiag}\{A_j^2\}_{j=1}^{m_d}, \quad B^2 = \text{blkdiag}\{B_j^2\}_{j=1}^{m_d}, \quad C^2 = \text{blkdiag}\{C_j^2\}_{j=1}^{m_d}, \quad D^2 = \text{blkdiag}\{D_j^2\}_{j=1}^{m_d}.$$

B. PROOF OF LEMMA 1

We will prove Lemma 1 by explicit construction of the precompensator (7) for each agent. The design is carried in three steps.

Step 1: Squaring-down precompensator

In this step, we design a compensator for each agent $i \in \{1, \dots, N\}$ such that the resulting system is invertible. Since the triple (C_i, A_i, B_i) is right-invertible, in order to do so, we only need to design a pre-compensator of the form:

$$\begin{cases} \dot{\chi}_i^1 = A_i^1 \chi_i^1 + B_i^1 u_i^1, \\ u_i = C_i^1 \chi_i^1 + D_i^1 u_i^1, \end{cases} \quad (50)$$

where $u_i^1 \in \mathbb{R}^p$, such that the resulting system of (5) and (50) is invertible. The design procedure was developed in [22] and reviewed in Appendix A.2.

Step 2: Rank-equalizing precompensator

It is clear that the resulting system of (5) and (50) is invertible. For a given $n_q \geq \bar{n}_d$, where $\bar{n}_d$ is the maximal order of infinite zero of (C_i, A_i, B_i) for all $i = 1, \dots, N$, we design a *rank-equalizing* precompensator of the form

$$\begin{cases} \dot{\chi}_i^2 = A_i^2 \chi_i^2 + B_i^2 u_i^2, \\ u_i^1 = C_i^1 \chi_i^2 + D_i^1 u_i^2, \end{cases} \quad (51)$$

where $u_i^2 \in \mathbb{R}^p$, such that the resulting system of (5), (50) and (51) is invertible and has uniform rank n_q . The design procedure was developed in [20] and reviewed in Appendix A.3.

Step 3: Observer-based pre-feedback

The third stage is to design an observer-based controller such that the resulting system is given by (8) and (9).

It is clear that the resulting system of (5), (50) and (51) is invertible and has uniform rank n_q . It is easy to see that there exists a nonsingular state transformation $\tilde{\Gamma}_{i,s}$ such that

$$\begin{bmatrix} x_i \\ \chi_i^1 \\ \chi_i^2 \end{bmatrix} = \tilde{\Gamma}_{i,s} \tilde{\chi}_i, \quad \tilde{\chi}_i = \begin{bmatrix} \tilde{\chi}_{i,a} \\ \tilde{\chi}_{i,d} \end{bmatrix},$$

such that the resulting system of (5), (50) and (51) can be written in the SCB form:

$$\begin{cases} \dot{\tilde{\chi}}_{i,a} = \tilde{A}_{i,a}\tilde{\chi}_{i,a} + \tilde{L}_{i,ad}y_i, \\ \dot{\tilde{\chi}}_{i,d} = \tilde{A}_d\tilde{\chi}_{i,d} + \tilde{B}_d(u_i^2 + D_{i,a}\tilde{\chi}_{i,a} + D_{i,d}\tilde{\chi}_{i,d}), \\ y_i = \tilde{C}_d\tilde{\chi}_{i,d}, \end{cases} \quad (52)$$

where

$$\tilde{A}_d = \begin{bmatrix} 0 & I_{p(n_q-1)} \\ 0 & 0 \end{bmatrix}, \quad \tilde{B}_d = \begin{bmatrix} 0 \\ I_p \end{bmatrix}, \quad \tilde{C}_d = [I_p \quad 0].$$

Note that the information $\tilde{z}_i := [z_i; \chi_i^1; \chi_i^2]$ is available for agent i , and $\tilde{z}_i$ can be represented in terms of $\tilde{\chi}_{i,a}$ and $\tilde{\chi}_{i,d}$ as:

$$\tilde{z}_i = \tilde{C}_i \begin{bmatrix} \tilde{\chi}_{i,a} \\ \tilde{\chi}_{i,d} \end{bmatrix},$$

where

$$\tilde{C}_i = \begin{bmatrix} C_i^m & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \tilde{\Gamma}_{i,s}.$$

Define

$$\tilde{A}_i = \begin{bmatrix} \tilde{A}_{i,a} & \tilde{L}_{i,ad}\tilde{C}_d \\ \tilde{B}_d D_{i,a} & \tilde{A}_d + \tilde{B}_d D_{i,d} \end{bmatrix}, \quad \tilde{B}_i = \begin{bmatrix} 0 \\ \tilde{B}_d \end{bmatrix}$$

It is clear that $(\tilde{C}_i, \tilde{A}_i)$ is detectable which follows from the fact that (C_i^m, A_i) is detectable. We then design the observer-based pre-feedback for the system (52) as:

$$\begin{cases} \dot{\hat{\chi}}_i = \tilde{A}_i\hat{\chi}_i + \tilde{B}_i v_i - \tilde{K}_i(\tilde{z}_i - \tilde{C}_i\hat{\chi}_i) \\ u_i^2 = [-D_{i,a} \quad \tilde{F}_d - D_{i,d}] \hat{\chi}_i + v_i, \end{cases} \quad (53)$$

where $v_i \in \mathbb{R}^p$ is a new input which will be designed in Section 3.3, $\tilde{K}_i$ is such that $\tilde{A}_i + \tilde{K}_i\tilde{C}_i$ is Hurwitz stable, and $\tilde{F}_d$ is such that $\tilde{A}_d + \tilde{B}_d\tilde{F}_d$ has desired eigenvalues. It is easy then to see that the observer error dynamics $\omega_i = \tilde{\chi}_i - \hat{\chi}_i$ is asymptotically stable, therefore, the injection term $\tilde{\chi}_{i,a}$ into the dynamics $\tilde{\chi}_{i,d}$ is asymptotically canceled. Hence, the mapping from v_i to y_i is given by

$$\begin{cases} \dot{\tilde{x}}_{i,d} = (\tilde{A}_d + \tilde{B}_d\tilde{F}_d)\tilde{\chi}_{i,d} + \tilde{B}_d(v_i + \rho_i), \\ y_i = \tilde{C}_d\tilde{\chi}_{i,d}, \end{cases} \quad (54)$$

where

$$\begin{cases} \dot{\omega}_i = (\tilde{A}_i + \tilde{K}_i\tilde{C}_i)\omega_i, \\ \rho_i = [D_{i,a} \quad D_{i,d} - \tilde{F}_d] \omega_i. \end{cases} \quad (55)$$

It is clear that the system (54) is invertible, of uniform rank n_q , and has no invariant zero. Moreover, the system (54) is of the form (8) with $\tilde{x}_i := \tilde{x}_{i,d}$, the parameters

$$A = \tilde{A}_d + \tilde{B}_d\tilde{F}_d, \quad B = \tilde{B}_d, \quad C = \tilde{C}_d,$$

and (55) is of the form (9) with $A_{i,s} = \tilde{A}_i + \tilde{K}_i\tilde{C}_i$ and $C_{i,s} = [D_{i,a} \quad D_{i,d} - \tilde{F}_d]$.

Note that the observer-based pre-feedback (53) for the system in the original coordinate $[x_i; \chi_i^1; \chi_i^2]$ can be written as

$$\begin{cases} \dot{\hat{\chi}}_i = \tilde{A}_i\hat{\chi}_i + \tilde{B}_i v_i - \tilde{K}_i(\tilde{z}_i - \tilde{C}_i\hat{\chi}_i), \\ u_i^2 = [-D_{i,a} \quad \tilde{F}_d - D_{i,d}] \tilde{\Gamma}_{i,s}\hat{\chi}_i + v_i. \end{cases} \quad (56)$$

It is easy to see that the composition of (50), (51), and (56), yields a pre-compensator of the form (7) with the parameters defined in obvious ways.

C. MANIPULATION OF EXOSYSTEM

Consider an arbitrary exosystem given by

$$\begin{cases} \dot{x} = Ax, & x(0) = x_0, \\ y = Cx, \end{cases} \quad (57)$$

where $x \in \mathbb{R}^r$, $y \in \mathbb{R}^p$, (C, A) is observable, and C is full column rank.

From [2, Theorem 4.3.1], we know that there exist nonsingular transformations $T_s \in \mathbb{R}^{r \times r}$ and $T_o \in \mathbb{R}^{p \times p}$, such that, in the transformed state and output, $x = T_s \tilde{x}$, $y = T_o \tilde{y}$, where

$$\tilde{x} = \begin{bmatrix} \tilde{x}_1 \\ \vdots \\ \tilde{x}_p \end{bmatrix}, \quad \tilde{x}_i = \begin{bmatrix} \tilde{x}_{i,1} \\ \vdots \\ \tilde{x}_{i,k_i} \end{bmatrix}, \quad i = 1, \dots, p, \quad \tilde{y} = \begin{pmatrix} \tilde{y}_1 \\ \vdots \\ \tilde{y}_p \end{pmatrix},$$

we have

$$\begin{cases} \dot{\tilde{x}}_i = A_i \tilde{x}_i + L_i \tilde{y}, \\ \tilde{y}_i = [1 \ 0] \tilde{x}_i, \quad i = 1, \dots, p, \end{cases} \quad (58)$$

with an initial condition $\tilde{x}_i(0)$ related to $x(0)$ in an obvious way, L_i is a constant matrix of an appropriate dimension and

$$A_i = \begin{bmatrix} 0 & I_{k_i-1} \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{k_i \times k_i}$$

The set of integers $\{k_1, k_2, \dots, k_p\}$ is the observability index of (C, A) . Note that k_i for $i = 1, \dots, p$ are in general different. In order for the system to have uniform rank n_q , we then add an appropriate number of integrators to the bottom of each chain. In particular, define

$$\bar{x} = \begin{bmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_p \end{bmatrix}, \quad \bar{x}_i = \begin{bmatrix} \tilde{x}_i \\ x_i^2 \end{bmatrix} \in \mathbb{R}^{n_q}, \quad i = 1, \dots, p, \quad x_i^2(0) = 0.$$

we then obtain that

$$\begin{cases} \dot{\bar{x}}_i = \bar{A}_i \bar{x}_i + \bar{L}_i \tilde{y}, & \bar{x}_i(0) = [\tilde{x}_i(0); 0], \\ \tilde{y}_i = \bar{C}_i \bar{x}_i, & i = 1, \dots, p, \end{cases} \quad (59)$$

where

$$\bar{A}_i = \begin{bmatrix} 0 & I_{n_q-1} \\ 0 & 0 \end{bmatrix}, \quad \bar{L}_i = \begin{bmatrix} L_i \\ 0 \end{bmatrix}, \quad \bar{C}_i = [1 \ 0 \ \dots \ 0].$$

It is easy to see that the system (58) and the system (59) generate exactly the same output $\tilde{y}$. The system (58) can be rewritten in a more compact form as follows:

$$\begin{cases} \dot{\bar{x}} = \bar{A} \bar{x}, & \bar{x}(0) = [\bar{x}_1(0); \dots; \bar{x}_p(0)], \\ \tilde{y} = \bar{C} \bar{x}, \end{cases} \quad (60)$$

where

$$\bar{A} = \begin{bmatrix} \bar{A}_1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \bar{A}_p \end{bmatrix} + \begin{bmatrix} \bar{L}_1 \\ \vdots \\ \bar{L}_p \end{bmatrix} \bar{C}, \quad \bar{C} = \begin{bmatrix} \bar{C}_1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \bar{C}_p \end{bmatrix}$$

and where $\star$ represents a matrix of less interest, generates the same output as (57). Note that the eigenvalues of the matrix $\bar{A}$ consists of all the eigenvalues of A and additional zero eigenvalues, which are degenerate.

Next, let us define

$$\bar{B} = \begin{bmatrix} \bar{B}_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \bar{B}_p \end{bmatrix} \quad \text{where} \quad \bar{B}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^{n_q}.$$

It is then easy to see that $(\bar{C}, \bar{A}, \bar{B})$ is invertible, of uniform rank n_q , and has no invariant zero.

We then restore the output transformation T_o back to the system (60) as follows:

$$\begin{cases} \dot{\bar{x}} = \bar{A}\bar{x}, & \bar{x}(0) = [\bar{x}_1(0); \cdots; \bar{x}_p(0)], \\ y = T_o \bar{C} \bar{x}. \end{cases} \quad (61)$$

Note that the system (61) generate the same output as (57). Since the nonsingular output transformation does not change the zero structure and invertibility of the system, the system $(T_o \bar{C}, \bar{A}, \bar{B})$ is also invertible, of uniform rank n_q , and has no invariant zero. Finally, there exist a nonsingular state transformation that transforms the system (61) into the form of (10).