

Simultaneous external and internal stabilization of linear systems with input saturation and non-input-additive sustained disturbances

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Abstract

In this paper we study simultaneous external and internal stabilization of linear system under input saturation and non-input additive sustained disturbances. For systems that are asymptotic null controllable with bounded control, it is shown that a nonlinear dynamic feedback controller can be designed so that the closed-loop states remain bounded for any initial condition and for two classes of sustained disturbances, and that the equilibrium in the absence of disturbances is globally asymptotically stable.

Key words: Simultaneous stabilization, input saturation, low-and-high gain feedback.

1 Introduction

All physical systems operate under a variety of constraints. One of the ubiquitous constraints is actuator saturation. In this paper, we study simultaneous external and internal stabilization of linear systems under actuator saturation and non-input-additive sustained disturbances. To be precise, we define the notion of external stability as having bounded closed-loop states for all initial conditions and internal stability as global asymptotic stability of the origin of the closed-loop system in the absence of disturbances.

The study on stabilization of linear system subject to input saturation has a long history. It is prudent to first review the literature with respect to previous results in this area. Fuller (1969) pioneered in early research on internal stabilization by showing that a chain of integrators of order greater than or equal to 3 with input saturation can only be globally asymptotically stabilized by a nonlinear control law. Sontag & Sussmann (1990); Sussmann & Yang (1991); Yang (1993); Yang et al. (1997) showed that the global asymptotic

stabilization of linear system subject to input saturation is possible if and only if the open-loop system is *asymptotic null controllable with bounded input* (ANCBC). Moreover, such a stabilization in general requires nonlinear control law. These works usher in a research boom over two decades on internal stabilization of linear system subject to input saturation. Many important results have been reported in the literature (see Bernstein & Michel, 1995; Saberi & Stoorvogel, 1999; Tarbouriech & Garcia, 1997; Saberi et al., 2000b; Hu & Lin, 2001; Kapila & Grigoriadis, 2002, and references therein).

After these achievements in the internal stabilization, researchers have focused on the simultaneous external and internal stabilization problem. In the literature dealing with external stability of linear systems subject to input saturation, the types of disturbances studied have been classified as input-additive and non-input-additive. For the input-additive case, Hou et al. (1998); Saberi et al. (2000a) and Wang et al. (2011c) showed that simultaneous internal and \mathcal{L}_p (continuous-time) or ℓ_p (discrete-time) stabilization with finite gain for $p \in [1, \infty]$ can be achieved by a nonlinear low-and-high gain state feedback. The design of such a controller relies heavily on characterizing the solution of algebraic Riccati equation (ARE) (see Megretski, 1996; Lin, 1998). In a special case of open-loop neutrally stable systems, it has been shown by Liu et al. (1996), Bao et al. (2000) and Chitour & Lin (2003) that a linear state feedback achieves \mathcal{L}_p and ℓ_p stabilization with finite gain for $p \in [1, \infty]$ while rendering the origin globally asymptotically stable without disturbances. On the other hand, Stoorvogel et al. (1999)

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studied the non-input-additive case and found that \mathcal{L}_p and ℓ_p stabilization with finite gain are impossible, but \mathcal{L}_p and ℓ_p stabilization without finite gain are always attainable via a dynamic low-gain feedback. Moreover, for an open-loop neutrally stable system, it is attainable via a linear static state feedback (see Shi et al., 2003). Nevertheless, these results only apply to \mathcal{L}_p and ℓ_p disturbances for $p \in [1, \infty)$ (i.e., disturbances whose “energy” vanishes asymptotically), and not to sustained signals belonging to \mathcal{L}_∞ and ℓ_∞ .

For sustained signals that are non-input-additive, clearly not all disturbances can be managed appropriately as, for instance, a large constant disturbance aligned with the input could overpower the saturated control and lead to unbounded states. In view of this, Stoorvogel et al. (2011) proved that for disturbances that are exactly matched to the input and have magnitude smaller than the level of saturation by a known margin, a nonlinear static state or dynamic measurement feedback can be constructed to ensure a bounded closed-loop state. Of particular interest in dealing with sustained non-input-additive disturbances, is the study on identifying classes of disturbances for which a controller can be designed to yield bounded closed-loop state trajectories. Work along this line has been carried out by Wen et al. (2008) and Wang et al. (2012, 2011b,a). In this body of work, a set of disturbances has been defined consisting of all continuous- or discrete-time signals that do not contain large sustained frequency components corresponding to the system’s eigenvalues on the imaginary axis (continuous-time) or on the unit circle (discrete-time) of the complex plane. It was shown in Wang et al. (2012, 2011b) that for continuous-time double integrators and chain of integrators this class of disturbances can be handled by an appropriately chosen control law if they are matched with the input. Moreover, the same control law can also cope with any bounded disturbances that are misaligned with the input. Later on, Wang et al. (2011a) showed that for both continuous- and discrete-time open-loop neutrally stable systems controlled by a properly chosen linear feedback under saturation, this class of disturbances also leaves the closed-loop states bounded for any initial conditions. However, it was illustrated by an example that the disturbances containing large sustained components at frequencies corresponding to the open-loop system’s eigenvalues may lead to unbounded states regardless of the choice for the controller.

In this paper, based on the construction for a chain of integrator and for neutrally stable systems, we shall extend the results in Wang et al. (2012, 2011b,a) to general ANCBC systems which may have non-zero degenerate eigenvalues on the imaginary axis (continuous-time) or on the unit circle (discrete-time); in other words, systems that are at most critically unstable. It will be shown that the same class of disturbances identified in Wang et al. (2011a) can be tackled by a properly designed feedback controller. At the same time, the resulting closed-loop system in the absence of disturbances is globally asymptotically stable.

The paper is organized as follows: In the preliminaries section, we first recall some standard notations. Then we shall

define the system and the problem to be studied in the paper and make several necessary assumptions. Next, a special Jordan decomposition is introduced which is instrumental in establishing our results. A special class of disturbances is introduced in Section 3. After these preparations, we present the main results of this paper and its proof in Section 4. Finally, some technical results used in this paper are given in the appendix.

2 Preliminaries

2.1 Notation

We first recall some standard notations. \mathbb{C}^s denotes closed left half plane (continuous-time) and closed unit disk (discrete-time). \mathbb{C}^b denotes the imaginary axis for continuous-time system and the unit circle for discrete-time system. For $x \in \mathbb{R}^n$, $\|x\|$ denotes its Euclidean norm and x' denotes the transpose of x . For $X \in \mathbb{R}^{n \times m}$, $\|X\|$ denotes its induced 2-norm and X' denotes the transpose of X . For continuous-time (discrete-time) signal y , $\|y\|_\infty$ denotes its \mathcal{L}_∞ (ℓ_∞) norm. $\mathcal{L}_\infty(\delta)$ ($\ell_\infty(\delta)$) represent a set of continuous-time (discrete-time) signals whose \mathcal{L}_∞ (ℓ_∞) norm is less than δ .

2.2 Formulation

Consider the following system

$$\rho x = Ax + B\sigma(u) + Ed, \quad x(0) = x_0, \quad (1)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $d \in \mathbb{R}^p$. ρx denotes the derivative $\rho x = \dot{x}$ for continuous-time systems and the shift operator $(\rho x)(t) = x(t+1)$ for discrete-time systems. $\sigma(\cdot)$ is the standard saturation function, i.e. for $s \in \mathbb{R}^m$

$$\sigma(s) = \begin{bmatrix} \text{sign}(s_1) \min\{1, |s_1|\} \\ \vdots \\ \text{sign}(s_m) \min\{1, |s_m|\} \end{bmatrix}$$

In this paper, we are interested in sustained disturbances, for which we assume in the first place that $d \in \mathcal{L}_\infty$ for continuous-time systems and $d \in \ell_\infty$ for discrete-time systems.

The problem we will study is to find a class of disturbances, say Ω , for which the simultaneous global \mathcal{L}_∞ or ℓ_∞ and global asymptotic stabilization problem is solvable, i.e. there exists a controller $u = f(x, t)$ possibly nonlinear and dynamic such that

- (1) in the absence of disturbances, the origin is globally asymptotically stable;
- (2) for $d \in \Omega$, the states of the closed-loop system remain bounded for $t \geq 0$.

Since the global asymptotic stabilization without disturbances is required, it is a classical result that the system needs to be asymptotically null controllable with bounded control, i.e.

- (1) (A, B) is stabilizable;
- (2) A has all its eigenvalues in \mathbb{C}^s .

Such a system can be decomposed into the following form:

$$\begin{bmatrix} \rho x_s \\ \rho x_u \end{bmatrix} = \begin{bmatrix} A_s & 0 \\ 0 & A_u \end{bmatrix} \begin{bmatrix} x_s \\ x_u \end{bmatrix} + \begin{bmatrix} B_s \\ B_u \end{bmatrix} \sigma(u) + \begin{bmatrix} E_s \\ E_u \end{bmatrix} d,$$

where A_s is asymptotically stable, A_u has all its eigenvalues on \mathbb{C}^b and (A_u, B_u) is controllable. Since A_s is stable and $\sigma(\cdot)$ and d are bounded, it follows that the x_s dynamics will remain bounded no matter what controller is used. Therefore, without loss of generality, we can ignore the asymptotically stable dynamics and assume in (1) that (A, B) is controllable and all the eigenvalues of A are on \mathbb{C}^b .

Under the above assumption, we consider a linear system with input saturation and disturbances:

$$\rho x = Ax + B\sigma(u) + Ed, \quad x(0) = x_0 \quad (2)$$

where (A, B) is controllable and A has all its eigenvalues on \mathbb{C}^b . Suppose the eigenvalues of A have q different Jordan block sizes denoted by n_1, \dots, n_q . Without loss of generality, we can assume $x = (x'_1, x'_2, \dots, x'_q)'$, and A, B are in the following form

$$A = \begin{bmatrix} \bar{A}_1 & 0 & \dots & 0 \\ 0 & \bar{A}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \bar{A}_q \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_q \end{bmatrix}, \quad E = \begin{bmatrix} E_1 \\ E_2 \\ \vdots \\ E_q \end{bmatrix} \quad (3)$$

where

$$x_i = \begin{bmatrix} x_{i,1} \\ x_{i,2} \\ \vdots \\ x_{i,n_i-1} \\ x_{i,n_i} \end{bmatrix}, \quad \bar{A}_i = \underbrace{\begin{bmatrix} A_i & I & 0 & \dots & 0 \\ 0 & A_i & I & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & A_i & I \\ 0 & \dots & \dots & 0 & A_i \end{bmatrix}}_{n_i \times n_i \text{ blocks}}, \quad (4)$$

$$B_i = \begin{bmatrix} B_{i,1} \\ B_{i,2} \\ \vdots \\ B_{i,n_i-1} \\ B_{i,n_i} \end{bmatrix}, \quad E_i = \begin{bmatrix} E_{i,1} \\ E_{i,2} \\ \vdots \\ E_{i,n_i-1} \\ E_{i,n_i} \end{bmatrix},$$

$x_{i,j} \in \mathbb{R}^{p_i}$ with $n = \sum_{i=1}^q n_i p_i$ and $A'_i + A_i = 0$ for continuous-time systems and $A'_i A_i = I$ for discrete-time

systems. Note that the above form can be obtained by assembling together in the real Jordan canonical form those blocks corresponding to eigenvalues with the same Jordan block size.

We say the disturbance d is *aligned* if $E_{i,n_i} \neq 0$ for some $i = 1, \dots, q$ and *misaligned* if $E_{i,n_i} = 0$ for all $i = 1, \dots, q$.

3 A special class of disturbances

We consider the following class of disturbances as defined in Wang et al. (2011a):

$$\Omega_\infty = \left\{ d \in \mathcal{L}_\infty \mid \exists M > 0, \text{ s. t. } \forall i \in 1, \dots, \ell, \right. \\ \left. \forall t_2 > t_1 > 0, \left\| \int_{t_1}^{t_2} d(t) e^{j\omega_i t} dt \right\| \leq M \right\}, \quad (5)$$

in continuous-time case and

$$\Omega_\infty = \left\{ d \in \ell_\infty \mid \exists M > 0, \text{ s. t. } \forall i \in 1, \dots, \ell, \right. \\ \left. \forall t_2 \geq t_1 \geq 0, \left\| \sum_{t=t_1}^{t_2} d(t) e^{j\omega_i t} \right\| \leq M \right\}, \quad (6)$$

in discrete-time case, where $j\omega_i$ (continuous-time) or $e^{j\omega_i}$ (discrete-time), $i \in 1, \dots, \ell$, represents the eigenvalues of A . Here we assume that the system has ℓ different eigenvalues (repeated eigenvalues are counted once).

The integral $\int_{t_1}^{t_2} d(t) e^{j\omega_i t} dt$ or summation $\sum_{t=t_1}^{t_2} d(t) e^{j\omega_i t}$ is easily recognized as the value at ω_i of the Fourier transform of the signal $d(t)$ truncated to $[t_1, t_2]$. The definition of Ω_∞ implies that this value must be uniformly bounded regardless of the choice of time interval. In practical terms, a signal that belongs to Ω_∞ is a signal that has no sustained frequency component at any of the frequencies ω_i , $i \in 1, \dots, \ell$.

To better motivate the definition of Ω_∞ and demonstrate its importance, we recall the following example by Wang et al. (2011a)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sigma(u) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} d, \quad (7)$$

where $d(t) = a \sin(t + \theta)$. This system is clearly in the form of (2), (3) and (4) and d contains a frequency component corresponding to the system's eigenvalues at $\pm j$. It was shown that for a relatively large a , states diverge to infinity for any initial condition and any controller. A similar example for discrete-time systems can also be constructed.

4 Main results

In this section, we shall show that for aligned disturbances which either belong to Ω_∞ and/or misaligned disturbances which belong to \mathcal{L}_∞ or ℓ_∞ , a controller can be designed such that the states of the closed-loop system remain bounded for any initial condition, at the same time the origin in the absence of disturbances is globally asymptotically stable. Moreover, we shall show that a small aligned disturbances which does not belong to Ω_∞ can also be tolerated.

Without loss of generality, for any critically unstable system with input saturation and non-input-additive disturbances as given by (2), (3) and (4), we can equivalently rewrite the system in the following form

$$\rho x = Ax + B\sigma(u) + \bar{E}_1 d_1 + \bar{E}_2 d_2 + \bar{E}_3 d_3, \quad (8)$$

with $x(0) = x_0$. In the above system, d_1 is misaligned and contain arbitrary disturbances that belong to \mathcal{L}_∞ (continuous-time) or ℓ_∞ (discrete-time), d_2 contains all aligned disturbances belonging to Ω_∞ and d_3 contains aligned disturbances which do not belong to Ω_∞ but are sufficiently small. The system data A and B are given by (3) and (4). The \bar{E}_1 , \bar{E}_2 and \bar{E}_3 are in the form

$$\bar{E}_1 = \begin{bmatrix} \bar{E}_{1,1} \\ \vdots \\ \bar{E}_{1,q-1} \\ \bar{E}_{1,q} \end{bmatrix}, \quad \bar{E}_{1,i} = \begin{bmatrix} E_{i,1} \\ \vdots \\ E_{i,n_i-1} \\ 0 \end{bmatrix} \quad (9)$$

and

$$\bar{E}_j = \begin{bmatrix} \bar{E}_{j,1} \\ \vdots \\ \bar{E}_{j,q-1} \\ \bar{E}_{j,q} \end{bmatrix}, \quad \bar{E}_{j,i} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ E_{i,n_i}^j \end{bmatrix}, \quad j = 2, 3. \quad (10)$$

Next we shall design a controller which solves the simultaneous external and internal stabilization problem. Let (A, B) satisfy the assumptions made in the preceding section and $P(\varepsilon) > 0$ be the solution to a Continuous Parametric Lyapunov Equation (CPLE)

$$A'P(\varepsilon) + P(\varepsilon)A - P(\varepsilon)BB'P(\varepsilon) + \varepsilon P(\varepsilon) = 0. \quad (11)$$

or a Discrete Parametric Lyapunov Equation (DPLE)

$$(1 - \varepsilon)P(\varepsilon) = A'P(\varepsilon)A - A'P(\varepsilon)B(B'P(\varepsilon)B + I)^{-1}B'P(\varepsilon)A. \quad (12)$$

with $\varepsilon \in (0, 0.9]$. The existence of the positive definite $P(\varepsilon)$ and its following properties were shown by Zhou et al. (2008, 2009).

- (1) $P(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$;
- (2) $\frac{dP(\varepsilon)}{d\varepsilon} > 0$ for $\varepsilon > 0$;
- (3) $P(\varepsilon)$ is rational in ε .

The special structure of \bar{E}_1 yields the following crucial technical lemma.

Lemma 1 *Let $P(\varepsilon)$ be the solution to CPLE (11) or DPLE (12) associated with A and B given by (3) and (4). For any matrix \bar{E}_1 in the form of (9), there exists M such that for $\varepsilon \in (0, 1]$*

$$\bar{E}_1' P(\varepsilon) \bar{E}_1 \leq M \varepsilon^2 I$$

Proof: See Appendix. ■

We will construct a low-gain dynamic state feedback controller. The controller as given below has q states that will transiently replace the evolution of the bottom states of each Jordan block \bar{A}_i in generating feedback input into the system.

$$\begin{cases} \rho \hat{x}_i = A_i \hat{x}_i + B_{i,n_i} \sigma(F(\varepsilon_a(\bar{x}))\bar{x}), \\ u = K(x_b - \hat{x}) + F(\varepsilon_a(\bar{x}))\bar{x}, \end{cases} \quad (13)$$

for $i = 1, \dots, q$ where $\hat{x} = [\hat{x}'_1, \hat{x}'_2, \dots, \hat{x}'_q]'$ and

$$x_b = \begin{bmatrix} x_{1,n_1} \\ x_{2,n_2} \\ \vdots \\ x_{q,n_q} \end{bmatrix}, \quad \bar{x} = \begin{bmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_{q-1} \\ \bar{x}_q \end{bmatrix}, \quad \bar{x}_i = \begin{bmatrix} x_{i,1} \\ \vdots \\ x_{i,n_i-1} \\ \hat{x}_i \end{bmatrix}.$$

Note that \bar{x} is the system state x with bottom state segment x_{i,n_i} of each Jordan block \bar{A}_i replaced by controller states \hat{x}_i . The feedback input is generated based on \bar{x} instead of x . As will become clear in the proof, the underlying idea behind (13) is that by utilizing the states of controller and the property of Ω_∞ , we will be able to convert some aligned disturbances affecting the bottom states into misaligned disturbances which turns out to be less restricted.

The parameter K can be chosen as

$$K = \begin{cases} -\hat{B}', & \text{continuous-time;} \\ -\kappa \hat{B}' \hat{A}, & \text{discrete-time.} \end{cases}$$

where κ satisfies $8\kappa B'B \leq I$ and

$$\hat{B} = \begin{bmatrix} B_{1,n_1} \\ B_{2,n_2} \\ \vdots \\ B_{q,n_q} \end{bmatrix}, \quad \hat{A} = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & A_q \end{bmatrix}.$$

The other feedback gain $F(\varepsilon_a(\bar{x}))$ can be designed as fol-

lows

$$F(\varepsilon) = \begin{cases} -B'P(\varepsilon), & \text{(continuous);} \\ -(B'P(\varepsilon)B + I)^{-1}B'P(\varepsilon)A, & \text{(discrete)} \end{cases}$$

where $P(\varepsilon)$ is the solution to CPLE (11) or DPLE (12). The parameter ε is determined by

$$\varepsilon = \varepsilon_a(\bar{x}) := \max\{r \in (0, 0.9] \mid (\bar{x}'P(r)\bar{x}) \text{trace}(P(r)) \leq \frac{\delta^2}{b}\} \quad (14)$$

where $b = 2 \text{trace}(BB')$, $\delta = \frac{1}{4}$ and $P(r)$ is the solution of (11) and (12) with $\varepsilon = r$. The scheduling (14) satisfies the following properties:

- (1) There exists an open neighborhood \mathcal{O} of the origin such that for all $\bar{x} \in \mathcal{O}$, $\varepsilon_a(\bar{x}) = 0.9$.
- (2) For any $\bar{x} \in \mathbb{R}^n$, $\|F(\varepsilon_a(\bar{x}))\bar{x}\| \leq \delta$.
- (3) $\varepsilon_a(\bar{x}) \rightarrow 0 \iff \|\bar{x}\| \rightarrow \infty$.
- (4) For each $c > 0$, the set $\{\bar{x} \in \mathbb{R}^n \mid \bar{x}'P(\varepsilon_a(\bar{x}))\bar{x} \leq c\}$ is bounded.
- (5) For any x_1 and x_2 , $x_1'P(\varepsilon_a(x_1))x_1 \leq x_2'P(\varepsilon_a(x_2))x_2 \implies \varepsilon_a(x_1) \geq \varepsilon_a(x_2)$.

(see Megretski, 1996; Lin, 1998; Hou et al., 1998; Wang et al., 2011c). The main result of this paper is stated in the following theorem:

Theorem 1 Consider the system (8) with controller (13). We have that

- (1) in the absence of d_1 , d_2 and d_3 , the origin is globally asymptotically stable;
- (2) there exists a $\delta_1 > 0$ such that the state remains bounded for any initial condition x_0 and disturbances $d_1 \in \mathcal{L}_\infty$, $d_2 \in \Omega_\infty$, $d_3 \in \mathcal{L}_\infty(\delta_1)$ (continuous time) or $d_1 \in \ell_\infty$, $d_2 \in \Omega_\infty$, $d_3 \in \ell_\infty(\delta_1)$ (discrete time).

Proof: We shall only prove the results for continuous-time systems. The discrete-time counterpart can be shown using a very similar argument. For continuous-time system, define

$$\tilde{x} = x_b - \hat{x} = \begin{bmatrix} x_{1,n_1} - \hat{x}_1 \\ x_{2,n_2} - \hat{x}_2 \\ \vdots \\ x_{q,n_q} - \hat{x}_q \end{bmatrix}.$$

We have that

$$\dot{\tilde{x}} = \hat{A}\tilde{x} + \hat{B}\sigma(-\hat{B}'\tilde{x} - B'P(\varepsilon_a(\bar{x}))\tilde{x}) - \hat{B}\sigma(-B'P(\varepsilon_a(\bar{x}))\tilde{x}) + \hat{E}_2d_2 + \hat{E}_3d_3,$$

where

$$\hat{E}_j = \begin{bmatrix} E_{1,n_1}^j \\ E_{2,n_2}^j \\ \vdots \\ E_{q,n_q}^j \end{bmatrix}, \quad j = 2, 3. \quad (15)$$

\hat{E}_2d_2 and \hat{E}_3d_3 contain all the aligned disturbances that affect the bottom states of each Jordan block \hat{A}_i . Note that (A, B) is controllable implies that (\hat{A}, \hat{B}) is controllable. Moreover, $\hat{A} + \hat{A}' = 0$. To simplify our presentation, we will denote $P(\varepsilon_a(\bar{x}))$ by P since the dependency on the scaling parameter should be clear from the context. The closed-loop system can be written in terms of \tilde{x} , \bar{x} as

$$\begin{cases} \dot{\tilde{x}} = A\bar{x} + B\sigma(-B'P\bar{x}) + \bar{E}_1d_1 + \mathcal{J}\tilde{x} \\ \quad + \bar{B}[\sigma(-\hat{B}'\tilde{x} - B'P\bar{x}) - \sigma(-B'P\bar{x})] \\ \dot{\tilde{x}} = \hat{A}\tilde{x} + \hat{B}\sigma(-\hat{B}'\tilde{x} - B'P\bar{x}) \\ \quad - \hat{B}\sigma(-B'P\bar{x}) + \hat{E}_2d_2 + \hat{E}_3d_3, \end{cases} \quad (16)$$

where \bar{B} is the same as B in (3) and (4) with B_{i,n_i} blocks set to zero and

$$\mathcal{J} = \begin{bmatrix} \mathcal{J}_1 \\ \mathcal{J}_2 \\ \vdots \\ \mathcal{J}_q \end{bmatrix}, \quad \mathcal{J}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \bar{I}_i \\ 0 \end{bmatrix}, \quad \bar{I}_i = [0 \quad \cdots \quad \underset{\substack{\uparrow \\ \text{i th block}}}{I} \quad \cdots \quad 0]$$

It should be noted that \bar{B} , \bar{E}_1 and \mathcal{J} are all in the form of (9). We first prove global asymptotic stability without disturbances. Consider the dynamics of \tilde{x} . Let $v = -B'P\bar{x}$. Our scheduling (14) guarantees that $\|v\| \leq \delta \leq \frac{1}{2}$ for any \bar{x} . Then,

$$\dot{\tilde{x}} = \hat{A}\tilde{x} + \hat{B}\sigma(-\hat{B}'\tilde{x} + v) - \hat{B}\sigma(v).$$

and define a Lyapunov function as $V_1 = \tilde{x}'\tilde{x}$. Differentiating V_1 along the trajectories yields

$$\dot{V}_1 = 2\tilde{x}'\hat{B}[\sigma(-\hat{B}'\tilde{x} + v) - \sigma(v)].$$

Since $\|v\| \leq \frac{1}{2}$, (18) yields that

$$\dot{V}_1 \leq -\tilde{x}'\hat{B}\sigma(\hat{B}'\tilde{x}).$$

Since \tilde{x} has a bounded derivative, by Barbalat's Lemma, this yields that $\lim_{t \rightarrow \infty} \hat{B}'\tilde{x}(t) = 0$ which implies that there exists T_0 such that we have $\|\hat{B}'\tilde{x}(t)\| \leq \frac{1}{2}$ for $t \geq T_0$ and hence

$$\dot{\tilde{x}} = (\hat{A} - \hat{B}\hat{B}')\tilde{x}$$

and since this system matrix is Hurwitz stable, we have $\tilde{x}(t) \rightarrow 0$ as $t \rightarrow \infty$. For $t > T_0$, we have that

$$\dot{\tilde{x}} = A\tilde{x} + B\sigma(-B'P\tilde{x}) + \bar{J}\tilde{x}$$

where $\bar{J} = J - \bar{B}\hat{B}'$. Define $V_2 = \tilde{x}'P\tilde{x}$ and a set

$$\mathcal{K} = \left\{ \tilde{x} \mid V_2(\tilde{x}) \leq \frac{\delta^2}{b \operatorname{trace}(P(0.9))} \right\}.$$

It can be easily seen from (14) that for $\tilde{x} \in \mathcal{K}$, $\varepsilon_a(\tilde{x}) = 0.9$. Next, consider the derivative of V_2 ,

$$\begin{aligned} \dot{V}_2 &= -\varepsilon V_2 - \tilde{x}'PBB'P\tilde{x} + 2\tilde{x}'P\bar{J}\tilde{x} + \tilde{x}'\frac{dP}{dt}\tilde{x} \\ &\leq -\varepsilon V_2 + 2\tilde{x}'P\bar{J}\tilde{x} + \tilde{x}'\frac{dP}{dt}\tilde{x} \\ &\leq -\varepsilon V_2 + 2\sqrt{V_2}\|P^{1/2}\bar{J}\tilde{x}\| + \tilde{x}'\frac{dP}{dt}\tilde{x}. \end{aligned}$$

Note that J , \bar{B} and hence \bar{J} are in the form of (9). Lemma 1 shows that there exists an M such that

$$\|P^{1/2}\bar{J}\tilde{x}\| = \sqrt{\tilde{x}'\bar{J}'P\bar{J}\tilde{x}} \leq \varepsilon\sqrt{M}\|\tilde{x}\|.$$

We use here that Lemma 1 holds for any matrix of the form (9) so it also holds for \bar{E}_1 replaced by \bar{J} . Hence

$$\begin{aligned} \dot{V}_2 &\leq -\varepsilon V_2 + 2\varepsilon\sqrt{M}\|\tilde{x}\|\sqrt{V_2} + \tilde{x}'\frac{dP}{dt}\tilde{x} \\ &\leq -\varepsilon\sqrt{V_2}\left[\sqrt{V_2} - 2\sqrt{M}\|\tilde{x}\|\right] + \tilde{x}'\frac{dP}{dt}\tilde{x}. \end{aligned}$$

Since $\tilde{x} \rightarrow 0$, there exists a $T_1 > T_0$ such that for $t \geq T_1$,

$$\|\tilde{x}\| \leq \frac{\delta}{4\sqrt{M}\sqrt{b \operatorname{trace}(P(0.9))}}.$$

Therefore, for $t \geq T_1$ and $\tilde{x} \notin \mathcal{K}$ we have

$$\sqrt{V_2} - 2\sqrt{M}\|\tilde{x}\| \geq \frac{\sqrt{V_2}}{2}$$

and thus

$$\dot{V}_2 \leq -\frac{\varepsilon}{2}V_2 + \tilde{x}'\frac{dP}{dt}\tilde{x}.$$

Since \dot{V}_2 cannot have the same sign as $\tilde{x}'\frac{dP}{dt}\tilde{x}$ (see Hou et al. (1998)), we conclude that $\dot{V}_2 < 0$ for $\tilde{x} \notin \mathcal{K}$ and $t > T_1$. This implies that \tilde{x} will enter \mathcal{K} within finite time, say $T_2 > T_1$, and remain in \mathcal{K} thereafter. For $t > T_2$ and $\tilde{x} \in \mathcal{K}$, we have $\varepsilon_a(\tilde{x}) = 0.9$ and $\|\hat{B}'\tilde{x}\| \leq \frac{1}{2}$. All saturations are inactive and the system becomes

$$\begin{cases} \dot{\tilde{x}} = (A - BB'P(0.9))\tilde{x} + \bar{J}\tilde{x}, \\ \dot{\hat{x}} = (\hat{A} - \hat{B}\hat{B}')\hat{x}. \end{cases}$$

The global asymptotic stability follows from the properties that $\hat{A} - \hat{B}\hat{B}'$ and $A - BB'P(0.9)$ are Hurwitz stable. We proceed to show the boundedness of trajectories in presence of d_1 and d_2 . Define

$$R = e^{\hat{A}'t} \quad \text{and} \quad y = R\hat{x}.$$

Note that since $\hat{A} + \hat{A}' = 0$, R defines a rotation matrix. Moreover, we have that $\dot{R} = -R\hat{A}$. We obtain that

$$\dot{y} = R\hat{B}\sigma(-\hat{B}'R'y + v) - R\hat{B}\sigma(v) + R\hat{E}_2d_2 + R\hat{E}_3d_3$$

with $y(0) = \tilde{x}_0$ where $v = -B'P\tilde{x}$. Let \bar{y} satisfy

$$\dot{\bar{y}} = R\hat{E}_2d_2, \quad \bar{y}(0) = \tilde{x}_0.$$

Since $d_2 \in \Omega_\infty$, we find that $\bar{y} \in \mathcal{L}_\infty$ (see Wang et al., 2011a). Define $\tilde{y} = y - \bar{y}$. Then

$$\dot{\tilde{y}} = R\hat{B}\sigma(-\hat{B}'R'\tilde{y} - \hat{B}'R'\bar{y} + v) - R\hat{B}\sigma(v) + R\hat{E}_3d_3$$

with $\tilde{y}(0) = 0$. Again define $z = R'\tilde{y}$. We get

$$\dot{z} = \hat{A}z + \hat{B}\sigma(-\hat{B}'z - \hat{B}'R'\bar{y} + v) - \hat{B}\sigma(v) + \hat{E}_3d_3, \quad z(0) = 0,$$

Consider an auxiliary system

$$\dot{w} = (\hat{A} + \hat{B}\hat{F})w + \hat{E}_3d_3, \quad w(0) = 0,$$

where \hat{F} is such that $\hat{A} + \hat{B}\hat{F}$ is Hurwitz stable. For a selected \hat{F} , let δ_1 be sufficiently small such that $\|d_3\|_{\mathcal{L}_\infty} \leq \delta_1$ implies that $\|\hat{F}w\|_{\mathcal{L}_\infty} \leq 1/4$.

Let $\xi = z - w$. We have that

$$\dot{\xi} = \hat{A}\xi + \hat{B}\sigma(-\hat{B}'\xi + u) - \hat{B}\sigma(v) - \hat{B}\hat{F}w, \quad \xi(0) = 0,$$

where $u = -\hat{B}'w - \hat{B}'R'\bar{y} + v$. Since $u \in \mathcal{L}_\infty$ and $\|\sigma(v) + \hat{F}w\|_{\mathcal{L}_\infty} \leq 1/4 + 1/4 = 1/2$, it follows from Lemma 3 in the appendix that $\xi \in \mathcal{L}_\infty$. This implies that $\tilde{x} \in \mathcal{L}_\infty$.

Consider the dynamics of \bar{x}

$$\dot{\bar{x}} = A\bar{x} + B\sigma(-B'P\bar{x}) + \bar{B}\zeta + \bar{E}_1d_1 + J\bar{x}$$

where $\zeta = \sigma(-\hat{B}'\tilde{x} - B'P\bar{x}) - \sigma(-B'P\bar{x})$. Since $\sigma(\cdot)$ is globally Lipschitz with Lipschitz constant 1, we have that $\|\zeta\| \leq \|\hat{B}'\tilde{x}\|$ and thus $\zeta \in \mathcal{L}_\infty$.

By differentiating $V_2 = \bar{x}'P\bar{x}$, we obtain

$$\begin{aligned} \dot{V}_2 &\leq -\varepsilon V_2 + 2\bar{x}'P\bar{E}_1d_1 + 2\bar{x}'PJ\bar{x} + 2\bar{x}'P\bar{B}\zeta + \bar{x}'\frac{dP}{dt}\bar{x} \\ &\leq -\varepsilon V_2 + 2\sqrt{V_2}\|P^{1/2}\bar{E}_1d_1\| + 2\sqrt{V_2}\|P^{1/2}\bar{B}\zeta\| \\ &\quad + 2\sqrt{V_2}\|P^{1/2}J\bar{x}\| + \bar{x}'\frac{dP}{dt}\bar{x}. \end{aligned}$$

We have already shown in Lemma 1 that there exist M_1 , M_2 and M_3 such that

$$\begin{aligned} \|P^{1/2}\bar{E}_1d_1\| &\leq \varepsilon\sqrt{M_1}\|d_1\|, \quad \|P^{1/2}\bar{B}\zeta\| \leq \varepsilon\sqrt{M_2}\|\zeta\| \\ \text{and } \|P^{1/2}J\bar{x}\| &\leq \varepsilon\sqrt{M_3}\|\bar{x}\|. \end{aligned}$$

We obtain,

$$\dot{V}_2 \leq -\varepsilon\sqrt{V_2} \left[\sqrt{V_2} - 2\sqrt{M_1}\|d_1\|_\infty - 2\sqrt{M_3}\|\bar{x}\|_\infty - 2\sqrt{M_2}\|\zeta\|_\infty \right] + \bar{x}' \frac{dP}{dt} \bar{x}.$$

If $\sqrt{V_2} \geq 2\sqrt{M_1}\|d_1\|_\infty + 2\sqrt{M_3}\|\bar{x}\|_\infty + 2\sqrt{M_2}\|\zeta\|_\infty$, we have

$$\dot{V}_2 \leq \bar{x}' \frac{dP}{dt} \bar{x}.$$

Since \dot{V}_2 and $\bar{x}' \frac{dP}{dt} \bar{x}$ can not have the same sign, we find that $\dot{V}_2 \leq 0$ for

$$\bar{x} \in \left\{ \bar{x} \mid \sqrt{V_2} \geq 2\sqrt{M_1}\|d_1\|_\infty + 2\sqrt{M_3}\|\bar{x}\|_\infty + 2\sqrt{M_2}\|\zeta\|_\infty \right\},$$

which, from the property (4) of scheduling (14), implies that $\bar{x} \in \mathcal{L}_\infty$ and hence $x \in \mathcal{L}_\infty$. ■

5 Computational issues

The proposed controller design relies on scheduling of the parameter $\varepsilon_a(\bar{x})$ which is a convex optimization problem but requires online solving CPLE (11) or DPLE (12) and can be computationally demanding for large systems. However, compared with normal Riccati equation, (11) and (12) still have some numerical merit, for example the solution $P(\varepsilon)$ is a rational matrix in general (see Zhou et al., 2008, 2009). Moreover, in a special case where the system has a single input, $P(\varepsilon)$ is a polynomial matrix and can be solved easily and explicitly in a finite recursion. In such a case, $\varepsilon_a(\bar{x})$ is not difficult to obtain.

Appendix

We shall need the following bounds from Shi et al. (2003):

Lemma 2 For two vectors $s, t \in \mathbb{R}^m$, the following statements hold:

$$\|s'[\sigma(s+t) - \sigma(s)]\| \leq 2\sqrt{m}\|t\|, \quad (17)$$

$$2s'[\sigma(t) - \sigma(t-s)] \geq s'\sigma(s), \quad \|t\| \leq \frac{1}{2}, \quad (18)$$

$$\|s - \sigma(s)\| \leq s'\sigma(s). \quad (19)$$

The following lemma is a core result for neutrally stable systems:

Lemma 3 Suppose (A, B) is controllable and $A' + A = 0$ for continuous-time systems and $A'A = I$ for discrete-time systems. Consider the system

$$\rho x = Ax + B\sigma(Kx + v_1) + Bv_2, \quad x(0) = x_0$$

where

$$K = \begin{cases} -B', & \text{continuous-time;} \\ -\kappa B'A, & \text{discrete-time.} \end{cases}$$

and κ satisfies $8\kappa B'B \leq I$. We have

- (1) In absence of v_1 and v_2 , the origin is globally asymptotically stable;
- (2) $x \in \mathcal{L}_\infty$ for any initial condition and for $v_1 \in \mathcal{L}_\infty$, $v_2 \in \mathcal{L}_\infty(1/2)$ (continuous time) or $v_1 \in \ell_\infty$, $v_2 \in \ell_\infty(1/2)$ (discrete time).

Proof: The result for continuous-time systems can be found in Liu et al. (1996, Lemma 2) and Yakoubi & Chitour (2006, Proposition 1). The discrete-time counterpart was proved by Wang et al. (2011a). ■

Proof of Lemma 1: We only prove the result for the continuous-time case. The corresponding discrete-time result follows from exactly the same argument. It is shown in (Zhou et al., 2008) that $P(0) = 0$ and $P(\varepsilon)$ is rational in ε . Therefore, we can write

$$P(\varepsilon) = \varepsilon P_1 + \varepsilon^2 P_2 + \dots + \varepsilon^i P_i + \dots$$

Substituting $P(\varepsilon)$ in (11), we find P_1 satisfies that

$$P_1 A + A' P_1 = 0, \quad (20)$$

where A is given by (3). Consider the diagonal block of P_1 , say $P_{1,i}$, corresponding to \bar{A}_i block. $P_{1,i}$ must satisfy

$$\bar{A}_i' P_{1,i} + P_{1,i} \bar{A}_i = 0 \quad (21)$$

where \bar{A}_i is given by (4). Suppose

$$P_{1,i} = \begin{bmatrix} \bar{P}_{11} & \bar{P}'_{12} \\ \bar{P}_{12} & \bar{P}_{22} \end{bmatrix}$$

where $\bar{P}_{11} \in \mathbb{R}^{p_i \times p_i}$, \bar{P}_{12} and \bar{P}_{22} are of appropriate dimension. Define

$$\xi_{i,j} = \begin{bmatrix} x'_{i,j} & 0 & \dots & 0 \end{bmatrix}'.$$

where the eigenvectors of A_i are $x_{i,j}$ with associated eigenvalues λ_j , $j = 1, \dots, p_i$. Clearly we have $\bar{A}_i \xi_{i,j} = \lambda_j \xi_{i,j}$ and thus $\xi_{i,j}$ is an eigenvector of \bar{A}_i . Note that \bar{A}_i has p_i linearly independent eigenvectors. We shall have that

$$(\bar{A}_i' P_{1,i} + \bar{P}_{1,i} \bar{A}_i) \xi_{i,j} = 0.$$

This implies that

$$\bar{A}_i' P_{1,i} \xi_{i,j} = -\lambda_j P_{1,i} \xi_{i,j}.$$

In other words, $P_{1,i} \xi_{i,j}$ is an eigenvector of \bar{A}_i' associated with eigenvalue $-\lambda_j$ for $j = 1, \dots, p_i$. On the other hand, we have a set of eigenvectors of \bar{A}' in the form of

$$v_{i,j} = \begin{bmatrix} 0 & \dots & 0 & v'_{i,j} \end{bmatrix}', \quad i = 1, \dots, p_i$$

where $v_{i,j}$ are the eigenvectors of A_i' associated with eigenvalue λ_j . Note that A_i' also has only p_i linearly independent eigenvectors. Therefore,

$$P_1 \xi_{i,j} = \begin{bmatrix} \bar{P}_{11} x_i \\ \bar{P}_{12} x_i \end{bmatrix} \in \text{span}\{v_{i,1}, \dots, v_{i,p_i}\}.$$

This implies that $\bar{P}_{11} x_{i,j} = 0$, $j = 1, \dots, p_i$. Since $x_{i,j}$ forms a basis of \mathbb{R}^{p_i} , we must have that $\bar{P}_{11} = 0$ and hence $\bar{P}_{12} = 0$ due to the fact that $P_{1,i}$ is positive semi-definite. Recursively, applying the above argument to \bar{P}_{22} , we shall eventually find that

$$P_{1,i} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & 0 & 0 \\ 0 & \dots & 0 & \bar{P}_{n_i n_i} \end{bmatrix}.$$

Since P_1 is positive semi-definite, we note that for any given matrix \bar{E}_1 in the form of (9), we must have $P_1 \bar{E}_1 = 0$. This implies that $\bar{E}_1' P(\varepsilon) \bar{E}_1$ must be of order ε^2 . ■

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