

Control of linear systems with input saturation and non-input-additive sustained disturbances – Continuous-time systems

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Abstract—In this paper we study control under input saturation and disturbances of neutrally stable and critically unstable linear systems. It is shown that a feedback controller can be designed so that the closed-loop states remain bounded for disturbances which belong to two different classes depending on the specific system structure and the equilibrium is globally asymptotically stable.

integral-bounded signals has been defined as

$$\mathcal{S}_\infty = \left\{ d \in \mathcal{L}_\infty \mid \exists M \text{ such that } \forall t_2 > t_1 > 0, \left\| \int_{t_1}^{t_2} d(t) dt \right\| < M \right\}.$$

I. INTRODUCTION

In this paper, we study the control of linear systems subject to input saturation and external disturbances.

In the literature dealing with external stability of linear systems subject to input saturation, the types of disturbances studied have been classified as input-additive and non-input-additive. For the input-additive case, simultaneous internal and \mathcal{L}_p stabilization with finite gain for $p \in [1, \infty]$ can be achieved by a nonlinear low-and-high gain state feedback [1], [4]. In a special case of open-loop neutrally stable systems, it has been shown that a linear state feedback achieves \mathcal{L}_p stability with finite gain for $p \in [1, \infty]$ [2]. On the other hand, \mathcal{L}_p stabilization with finite gain has been shown to be impossible in the non-input-additive case, but \mathcal{L}_p stabilization without finite gain is always attainable via a dynamic low-gain feedback [6]. Moreover, for an open-loop neutrally stable system, it is attainable via a linear state feedback [5]. Nevertheless, these results only apply to \mathcal{L}_p disturbances for $p \in [1, \infty)$ (i.e., disturbances whose “energy” vanishes asymptotically), and not to sustained signals belonging to \mathcal{L}_∞ .

For sustained signals that are non-additive, clearly all disturbances are not manageable as a large constant disturbance could dwarf the saturated control input and lead to unbounded states. One direction of research has focused on identifying classes of sustained disturbances for which a controller can be designed to yield bounded closed-loop state trajectories. Work along this line has been carried out by our group in [10], [9], [8], [7]. In that work, a set of

The set \mathcal{S}_∞ represents signals that have a uniformly bounded integral over every time interval; that is, signals that have no sustained DC bias. For neutrally stable systems consisting only of *single integrators* (i.e., simple eigenvalues at the origin), it has been shown that with a suitable choice of control input, the state trajectories remain bounded for all integral-bounded disturbances. Moreover, this result also holds if we add a sufficiently small signal that does not belong to \mathcal{S}_∞ .

The results for single-integrator systems appeared in the larger context of studying *chains of integrators*, for which it was shown in [8] that integral-bounded disturbances can be handled by an appropriately chosen control law if they are matched with the input. Moreover, the same control law can also cope with disturbances that are bounded and misaligned with input.

In this paper we shall first generalize the results for single-integrator systems to neutrally stable systems having complex eigenvalues on the imaginary axis. Roughly speaking, we shall show that for disturbances that do not have large sustained frequency components corresponding to the system’s eigenvalues on the imaginary axis, a linear static state feedback can be employed to achieve boundedness of the trajectories for any initial condition and at the same time yield a global asymptotically stable equilibrium.

Furthermore, based on the construction for neutrally stable systems, we shall extend results to general critically unstable systems which may have eigenvalues on the imaginary axis with Jordan block size greater than 1. It will be shown that disturbances, which may belong to two different classes depending on which state elements they directly corrupt in a special Jordan canonical form, can be handled by a properly designed feedback controller. At the same time, the resulting closed-loop system is globally asymptotically stable.

The following notations will be used in this paper. \mathbb{C}^s denotes closed left half plane. \mathbb{C}^b denotes the imaginary axis. For $x \in \mathbb{R}^n$, $\|x\|$ denotes its Euclidean norm and x' denotes the transpose of x . For $X \in \mathbb{R}^{n \times m}$, $\|X\|$ denotes its induced 2-norm and X' denotes the transpose of X . For continuous-time signal y , $\|y\|_\infty$ denotes its \mathcal{L}_∞ norm. $\mathcal{L}_\infty(\delta)$ represent

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a set of continuous-time signals whose \mathcal{L}_∞ norm is less than δ .

The paper is organized as follows: Preliminaries are given in Section II. Results for neutrally stable systems and critically unstable systems are respectively presented in Section III and IV. Some technical results used in the proof are given in the Appendix.

II. PRELIMINARIES

Consider the following system

$$\dot{x} = Ax + B\sigma(u) + Ed, \quad x(0) = x_0, \quad (1)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $d \in \mathbb{R}^p$ and $\sigma(\cdot)$ is the standard saturation function defined as $\sigma(s) = [\sigma_0(s_1); \dots; \sigma_0(s_m)]$ and $\sigma_0(s_i) = \text{sign}(s_i) \min\{1, |s_i|\}$.

In this paper, we are interested in sustained disturbances, for which we assume that $d \in \mathcal{L}_\infty$. We also use $\mathcal{L}_\infty(\delta)$ to denote the set of \mathcal{L}_∞ signals whose \mathcal{L}_∞ norm is less than δ . Since global asymptotic stabilization is required, it is a well-known result that system (1) has to be Asymptotically Null Controllable with Bounded Control (ANCBC), that is (A, B) is stabilizable and A has all its eigenvalues in the closed-left-half plane.

Moreover, the system is said to be *neutrally stable* if

- 1) A has all its eigenvalues in the closed-left-half plane;
- 2) A has at least one eigenvalue on the imaginary axis;
- 3) those eigenvalues on the imaginary axis are semi-simple.

The system is said to be *critically unstable* if

- 1) A has all its eigenvalues in the closed-left-half plane;
- 2) A has at least one eigenvalue on the imaginary axis which has Jordan block size greater than 1.

A system that is neutrally stable or critically unstable can be decomposed into the following form:

$$\begin{bmatrix} \dot{x}_s \\ \dot{x}_u \end{bmatrix} = \begin{bmatrix} A_s & \\ & A_u \end{bmatrix} \begin{bmatrix} x_s \\ x_u \end{bmatrix} + \begin{bmatrix} B_s \\ B_u \end{bmatrix} \sigma(u) + \begin{bmatrix} E_s \\ E_u \end{bmatrix} d,$$

where A_s is Hurwitz stable, A_u has all its eigenvalues on the imaginary axis and (A_u, B_u) is controllable. Since A_s is Hurwitz stable and $\sigma(\cdot)$ and d are bounded, it follows that the x_s dynamics will remain bounded no matter what controller is used. Therefore, without loss of generality, we can ignore the asymptotically stable dynamics and assume in (1) that (A, B) is controllable and all the eigenvalues of A are on the imaginary axis.

A. Extended class of disturbances

To present our results, we extend the definition of integral-bounded disturbances by defining a new set

$$\Omega_\infty = \left\{ d \in \mathcal{L}_\infty \mid \exists M \text{ s. t. } \forall i \in 1, \dots, \ell, \forall t_2 > t_1 > 0, \left\| \int_{t_1}^{t_2} d(t) e^{j\omega_i t} dt \right\| < M \right\}. \quad (2)$$

where $\pm j\omega_i$, $i \in 1, \dots, \ell$, represents the eigenvalues of A . Here we assume that the system has ℓ different eigenvalues (repeated eigenvalues are counted once). The set Ω_∞ consists of those signals that remain integral-bounded when multiplied by $\sin \omega_i t$ and $\cos \omega_i t$. This definition is a natural generalization of \mathcal{S}_∞ , since $\Omega_\infty = \mathcal{S}_\infty$ for $\omega_i = 0$.

The integral $\int_{t_1}^{t_2} d(t) e^{j\omega_i t} dt$ is easily recognized as the value at ω_i of the Fourier transform of the signal $d(t)$ truncated to the interval $[t_1, t_2]$. The definition of Ω_∞ implies that this value must be uniformly bounded regardless of the choice of t_1 and t_2 . In practical terms, a signal that belongs to Ω_∞ is a signal that has no sustained frequency component at any of the frequencies ω_i , $i \in 1, \dots, \ell$.

B. Preliminary results

We shall need the following inequalities, which were proven in [5]:

Lemma 1: For two vectors $s, t \in \mathbb{R}^m$, the following statements hold:

- 1) $|s'[\sigma(s+t) - \sigma(s)]| \leq 2\sqrt{m}\|t\|$
- 2) if $\|t\| \leq \frac{1}{2}$, then $2s'[\sigma(t) - \sigma(t-s)] \geq s'\sigma(s)$
- 3) $\|s - \sigma(s)\| \leq s'\sigma(s)$

III. NEUTRALLY STABLE SYSTEMS

We first consider the neutrally stable systems where A only has semi-simple eigenvalues on the imaginary axis. Without loss of generality, we can assume that $A + A' = 0$. We shall employ a linear static state feedback $u = -B'x$, which results in a closed-loop system

$$\dot{x} = Ax - B\sigma(B'x) + Ed, \quad x(0) = x_0. \quad (3)$$

It has been proved in [2] that the origin of (3) in the absence of d is globally asymptotically stable.

In keeping with the results for the single-integrator case, we shall show in what follows that the trajectories of the controlled system remain bounded for all disturbances belonging to Ω_∞ . Moreover, this result also holds if we add a sufficiently small signal that does not belong to Ω_∞ .

A. Single-frequency system

Now we are in position to present the main results of this paper. We start by considering an example system with a pair of complex eigenvalues at $\pm j$:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sigma(x_2) + \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} d. \quad (4)$$

Theorem 1: The trajectories of (4) remain bounded for any initial condition and $d \in \Omega_\infty$.

We shall only give an outline of the proof. To analyze the system, we introduce a rotation matrix

$$R = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}, \quad (5)$$

which represents a counterclockwise rotation by an angle t . The dynamics of the rotation matrix is given by

$$\dot{R} = -R \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

We can then study the dynamics of x from a rotated coordinate frame $y = Rx$. The detailed proof can be found in [7].

To demonstrate the importance of the disturbance belonging to Ω_∞ , we shall now show that if d contains a large frequency component at $\pm j$, the states of (4) will diverge toward infinity for any initial condition. Suppose therefore that $d(t) = a \sin(t + \theta)$, where a is an amplitude yet to be chosen. For ease of presentation, we assume that $(e_1, e_2)' = (0, 1)'$. Consider the dynamics of the rotated state $y = Rx$ where R is given by (5). We have

$$\begin{aligned} \dot{y} &= R \begin{bmatrix} 0 \\ 1 \end{bmatrix} (d - \sigma([0 \ 1] R'y)) \\ &= a \begin{bmatrix} -\sin t \sin(t + \theta) \\ \cos t \sin(t + \theta) \end{bmatrix} - R \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sigma(s(t)). \end{aligned}$$

where $s(t) = -\sin(t)y_1(t) + \cos(t)y_2(t)$. Using appropriate trigonometric identities, the dynamics can be rewritten as

$$\dot{y} = \frac{a}{2} \begin{bmatrix} \cos(2t + \theta) - \cos(-\theta) \\ \sin(2t + \theta) - \sin(-\theta) \end{bmatrix} - R \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sigma(s(t)).$$

We have that either $|\sin(\theta)| \geq \sqrt{2}/2$ or $|\cos(\theta)| \geq \sqrt{2}/2$. Without loss of generality, we assume that $|\sin(\theta)| \geq \sqrt{2}/2$. Let a be chosen such that $a \geq 4/\sqrt{2}(1 + \varepsilon)$, where ε is a positive number. For the trajectory $y_2(t)$, we have

$$\begin{aligned} |y_2(t)| &= \left| y_2(0) + \int_0^t \frac{a}{2} (\sin(2\tau + \theta) - \sin(-\theta)) \right. \\ &\quad \left. - [0 \ 1] R(\tau) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sigma(s(\tau)) d\tau \right|. \end{aligned}$$

Noting that the last term of the integrand is bounded by ± 1 , and using the bound $|a/2 \sin(\theta)| \geq \sqrt{2}a/4 \geq 1 + \varepsilon$, we therefore have

$$\begin{aligned} |y_2(t)| &\geq -|y_2(0)| - \frac{a}{2} \left| \int_0^t \sin(2\tau + \theta) d\tau \right| + \int_0^t \varepsilon d\tau \\ &\geq -|y_2(0)| - \frac{a}{2} + \varepsilon t. \end{aligned}$$

This shows that $y_2(t)$ diverges toward infinity.

B. Connection to the single-integrator case

Before moving on to the case of general multi-frequency neutrally stable systems, it is instructive to compare some aspects of the above example with single-integrator systems. A single-integrator system with a saturated control input and an external disturbance has the form

$$\dot{x} = \sigma(u) + ed.$$

In the absence of disturbances, the open-loop response of this system is stationary. It is intuitively easy to see that a large DC bias in d would constitute a problem, because it would tend to dominate the bounded control term $\sigma(\cdot)$, thus leading to divergence. The absence of such a DC bias is guaranteed by d belonging to \mathcal{S}_∞ .

The system with eigenvalues at $\pm j$ has the form

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sigma(\cdot) + \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} d.$$

In the absence of disturbances, the open-loop response of this system is oscillatory rather than stationary, and it is less obvious why a disturbance that does not belong to Ω_∞ may be problematic. By introducing a rotated state $y = Rx$, however, we obtain the dynamics

$$\dot{y} = R \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sigma(\cdot) + R \begin{bmatrix} 0 \\ 1 \end{bmatrix} d.$$

In the absence of disturbances, the open-loop response of y is stationary, and the dynamics of y are strikingly similar to the single-integrator case. In particular, it is easy to see that a large DC bias in the term $R \begin{bmatrix} 0 \\ 1 \end{bmatrix} d$ would constitute a problem, because it would tend to dominate the bounded control term. Analogous to the single-integrator case, the absence of such a bias is guaranteed if $R \begin{bmatrix} 0 \\ 1 \end{bmatrix} d$ belongs to \mathcal{S}_∞ , which is equivalent to d belonging to Ω_∞ .

In the single-integrator case, a DC bias in d can be tolerated if it is sufficiently small. Similarly, a small signal that does not belong to Ω_∞ can be tolerated for systems with complex eigenvalues. This is demonstrated in the next section, which deals with general multi-frequency systems.

C. Multi-frequency systems

We first extend Theorem 1 to a multi-frequency neutrally stable system. Consider

$$\dot{x} = Ax - B\sigma(B'x) + Ed, \quad x(0) = x_0 \quad (6)$$

where $A + A' = 0$ and (A, B) is controllable.

Theorem 2: The states of (6) remain bounded for any initial condition if $d \in \Omega_\infty$.

Moreover, a small disturbance which does not belong to Ω_∞ can also be tolerated. Consider the system

$$\dot{x} = Ax - B\sigma(B'x) + E_1 d_1 + E_2 d_2, \quad x(0) = x_0 \quad (7)$$

where $A + A' = 0$ and (A, B) is controllable.

Theorem 3: There exists a $\delta_1 > 0$ such that the states of system (7) remain bounded for all initial conditions if $d_1 \in \Omega_\infty$ and $\|d_2\|_{\mathcal{L}_\infty} \leq \delta_1$.

Theorem 2 and 3 can be proved by considering a higher dimensional rotation matrix $R = e^{A't}$ and a rotated coordinate frame $y = Rx$ and using Lemma 3 in the Appendix. The details are omitted here and can be found in [7].

IV. CRITICALLY UNSTABLE SYSTEMS

A. Formulation

Consider a linear system with input saturation and disturbances:

$$\dot{x} = Ax + B\sigma(u) + Ed, \quad x(0) = x_0 \quad (8)$$

where (A, B) is controllable and A has all its eigenvalues on the imaginary axis. Suppose the eigenvalues of A have q different Jordan block sizes denoted by n_1, \dots, n_q . Without

loss of generality, we can assume $x = (x'_1, x'_2, \dots, x'_q)'$, and A , B and E are in the following form

$$A = \begin{bmatrix} \bar{A}_1 & 0 & \cdots & 0 \\ 0 & \bar{A}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \bar{A}_q \end{bmatrix}, B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_q \end{bmatrix}, E = \begin{bmatrix} E_1 \\ E_2 \\ \vdots \\ E_q \end{bmatrix} \quad (9)$$

where

$$x_i = \begin{bmatrix} x_{i,1} \\ x_{i,2} \\ \vdots \\ x_{i,n_i-1} \\ x_{i,n_i} \end{bmatrix}, \bar{A}_i = \underbrace{\begin{bmatrix} A_i & I & 0 & \cdots & 0 \\ 0 & A_i & I & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & A_i & I \\ 0 & \cdots & \cdots & 0 & A_i \end{bmatrix}}_{n_i \times n_i \text{ blocks}},$$

$$B_i = \begin{bmatrix} B_{i,1} \\ B_{i,2} \\ \vdots \\ B_{i,n_i-1} \\ B_{i,n_i} \end{bmatrix}, E_i = \begin{bmatrix} E_{i,1} \\ E_{i,2} \\ \vdots \\ E_{i,n_i-1} \\ E_{i,n_i} \end{bmatrix}, \quad (10)$$

$x_{i,j} \in \mathbb{R}^{p_i}$ with $n = \sum_{i=1}^q n_i p_i$ and $A'_i + A_i = 0$. Note that the above form can be obtained by assembling together in the real Jordan canonical form those blocks corresponding to eigenvalues with the same Jordan block size.

We say the disturbance d is *aligned* if $E_{i,n_i} \neq 0$ for some $i = 1, \dots, q$ and *misaligned* if $E_{i,n_i} = 0$ for all $i = 1, \dots, q$.

Without loss of generality, for any critically unstable system with input saturation and non-input-additive disturbances as given by (8), (9) and (10), we can equivalently rewrite the system in the following form

$$\dot{x} = Ax + B\sigma(u) + \bar{E}_1 d_1 + \bar{E}_2 d_2 + \bar{E}_3 d_3, \quad (11)$$

with $x(0) = x_0$. In the above system, d_1 is misaligned and contains arbitrary disturbances that belong to \mathcal{L}_∞ , d_2 contains all aligned disturbances belonging to Ω_∞ and d_3 contains aligned disturbances which do not belong to Ω_∞ but are sufficiently small. The system data A and B are given by (9) and (10). The \bar{E}_1 , \bar{E}_2 and \bar{E}_3 are in the form

$$\bar{E}_1 = \begin{bmatrix} \bar{E}_{1,1} \\ \vdots \\ \bar{E}_{1,q-1} \\ \bar{E}_{1,q} \end{bmatrix}, \bar{E}_{1,i} = \begin{bmatrix} E_{i,1} \\ \vdots \\ E_{i,n_i-1} \\ 0 \end{bmatrix} \quad (12)$$

and

$$\bar{E}_j = \begin{bmatrix} \bar{E}_{j,1} \\ \vdots \\ \bar{E}_{j,q-1} \\ \bar{E}_{j,q} \end{bmatrix}, \bar{E}_{j,i} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ E_{i,n_i}^j \end{bmatrix}, \quad j = 2, 3. \quad (13)$$

B. Controller design

In parallel with the results for the chain of integrators, we shall show that if the disturbances are aligned and belong to Ω_∞ or misaligned and belong to \mathcal{L}_∞ , a controller can be designed such that the states of closed-loop system remain bounded for any initial condition while yielding a globally asymptotically stable equilibrium. For neutrally stable system, we have shown that this can be achieved by a linear state feedback. However, a nonlinear feedback controller is generally needed for critically unstable systems. In what follows, we present a nonlinear dynamic low-gain state feedback design methodology which solves our problem.

Let (A, B) satisfy the assumptions made in the preceding section and $P(\varepsilon) > 0$ be the solution to a Continuous Parametric Lyapunov Equation (CPLE)

$$A'P(\varepsilon) + P(\varepsilon)A - P(\varepsilon)BB'P(\varepsilon) + \varepsilon P(\varepsilon) = 0. \quad (14)$$

The following lemma is proven in Appendix.

Lemma 2: Let $P(\varepsilon)$ be the solution to CPLE (14) associated with A and B given by (9) and (10). For any matrix \bar{E}_1 in the form of (12), there exists M such that for $\varepsilon \in (0, 1]$

$$\bar{E}_1' P(\varepsilon) \bar{E}_1 \leq M \varepsilon^2 I$$

We will construct a low-gain dynamic state feedback controller. The controller as given below has q states that will transiently replace the evolution of the bottom states of each Jordan block \bar{A}_i in generating feedback input into the system.

$$\begin{cases} \dot{\hat{x}}_i = A_i \hat{x}_i + B_{i,n_i} \sigma(-B'P(\varepsilon)\bar{x}), \\ u = -\hat{B}'(x_b - \hat{x}) - B'P(\varepsilon)\bar{x}, \end{cases} \quad (15)$$

for $i = 1, \dots, q$ where,

$$\hat{B} = \begin{bmatrix} B_{1,n_1} \\ B_{2,n_2} \\ \vdots \\ B_{q,n_q} \end{bmatrix}, x_b = \begin{bmatrix} x_{1,n_1} \\ x_{2,n_2} \\ \vdots \\ x_{q,n_q} \end{bmatrix}, \hat{x} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \vdots \\ \hat{x}_q \end{bmatrix},$$

and

$$\bar{x} = \begin{bmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_{q-1} \\ \bar{x}_q \end{bmatrix}, \bar{x}_i = \begin{bmatrix} x_{i,1} \\ \vdots \\ x_{i,n_i-1} \\ \hat{x}_i \end{bmatrix}.$$

Note that \bar{x} is the system state x with bottom state segment x_{i,n_i} of each Jordan block \bar{A}_i replaced by controller states \hat{x}_i . The feedback input is generated based on \bar{x} instead of x . As will become clear in the proof, the underlying idea behind (15) is that by utilizing the states of controller and the property of Ω_∞ , we will be able to convert some aligned disturbances affecting the bottom states into misaligned disturbances which turns out to be less restricted.

Let $P(\varepsilon)$ be the solution to CPLE (14) and let ε be determined by

$$\varepsilon = \varepsilon_a(\bar{x}) := \max\{r \in [0, 1] \mid (\bar{x}'P(r)\bar{x}) \text{trace}(B'P(r)B) \leq \delta^2\} \quad (16)$$

where $\delta = \frac{1}{4}$. It has been shown in [3] that scheduling (16) guarantees that $\|\hat{B}'P(\varepsilon_a(\bar{x}))\bar{x}\| \leq \delta$ for all \bar{x} . Note that the scheduling of the parameter $\varepsilon_a(\bar{x})$ is a convex optimization problem but requires online solving CPLE (14) and can be computationally demanding for large systems. In a special case where the system has a single input, $P(\varepsilon)$ is a polynomial matrix and can be solved easily and explicitly in a finite recursion. In such a case, $\varepsilon_a(\bar{x})$ is not difficult to obtain (see [11])

C. Main results for critically unstable systems

Theorem 4: Consider the system (11) with controller (15). We have that

- 1) in the absence of d_1 , d_2 and d_3 , the origin is globally asymptotically stable;
- 2) there exists a $\delta_1 > 0$ such that the state remains bounded for any initial condition x_0 and disturbances $d_1 \in \mathcal{L}_\infty$, $d_2 \in \Omega_\infty$, $d_3 \in \mathcal{L}_\infty(\delta_1)$ (continuous time).

Proof: Define

$$\tilde{x} = x_b - \hat{x} = \begin{bmatrix} x_{1,n_1} - \hat{x}_1 \\ x_{2,n_2} - \hat{x}_2 \\ \vdots \\ x_{q,n_q} - \hat{x}_q \end{bmatrix}.$$

We have that

$$\begin{aligned} \dot{\tilde{x}} = & \hat{A}\tilde{x} + \hat{B}\sigma(-\hat{B}'\tilde{x} - B'P(\varepsilon_a(\bar{x}))\bar{x}) \\ & - \hat{B}\sigma(-B'P(\varepsilon_a(\bar{x}))\bar{x}) + \hat{E}_2d_2 + \hat{E}_3d_3, \end{aligned}$$

where

$$\hat{A} = \begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_q \end{bmatrix}, \quad \hat{E}_j = \begin{bmatrix} E_{1,n_1}^j \\ E_{2,n_2}^j \\ \vdots \\ E_{q,n_q}^j \end{bmatrix}, \quad j = 2, 3. \quad (17)$$

\hat{E}_2d_2 and \hat{E}_3d_3 contain all the aligned disturbances that affect the bottom states of each Jordan block \bar{A}_i . Note that (A, B) is controllable implies that (\hat{A}, \hat{B}) is controllable. Moreover, $\hat{A} + \hat{A}' = 0$. To simplify our presentation, we will denote $P(\varepsilon_a(\bar{x}))$ by P since the dependency on the scaling parameter should be clear from the context. The closed-loop system can be written in terms of \tilde{x}, \bar{x} as

$$\begin{cases} \dot{\tilde{x}} = A\bar{x} + B\sigma(-B'P\bar{x}) + \bar{E}_1d_1 + \mathcal{I}\tilde{x} \\ \quad + \bar{B}[\sigma(-\hat{B}'\tilde{x} - B'P\bar{x}) - \sigma(-B'P\bar{x})] \\ \dot{\bar{x}} = \hat{A}\bar{x} + \hat{B}\sigma(-\hat{B}'\tilde{x} - B'P\bar{x}) \\ \quad - \hat{B}\sigma(-B'P\bar{x}) + \hat{E}_2d_2 + \hat{E}_3d_3, \end{cases} \quad (18)$$

where \bar{B} is the same as B in (9) and (10) with B_{i,n_i} blocks set to zero and

$$\mathcal{I} = \begin{bmatrix} \mathcal{I}_1 \\ \mathcal{I}_2 \\ \vdots \\ \mathcal{I}_q \end{bmatrix}, \quad \mathcal{I}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \bar{I}_i \\ 0 \end{bmatrix}, \quad \bar{I}_i = [0 \quad \cdots \quad \underset{\substack{\uparrow \\ \text{i th block}}}{I} \quad \cdots \quad 0]$$

It should be noted that \bar{B} , \bar{E}_1 and \mathcal{I} are all in the form of (12). We first prove global asymptotic stability without disturbances. Consider the dynamics of \tilde{x} . Let $v = -B'P\bar{x}$. Our scheduling (16) guarantees that $\|v\| \leq \delta \leq \frac{1}{2}$ for any \bar{x} . Then,

$$\dot{\tilde{x}} = \hat{A}\tilde{x} + \hat{B}\sigma(-\hat{B}'\tilde{x} + v) - \hat{B}\sigma(v).$$

and define a Lyapunov function as $V_1 = \tilde{x}'\tilde{x}$. Differentiating V_1 along the trajectories yields

$$\dot{V}_1 = 2\tilde{x}'\hat{B}[\sigma(-\hat{B}'\tilde{x} + v) - \sigma(v)].$$

Since $\|v\| \leq \frac{1}{2}$, we have that

$$\dot{V}_1 \leq -\tilde{x}'\hat{B}\sigma(\hat{B}'\tilde{x}).$$

Since \tilde{x} has a bounded derivative, by Barbalat's Lemma, this yields that $\lim_{t \rightarrow \infty} \hat{B}'\tilde{x}(t) = 0$ which implies that there exists T_0 such that we have $\|\hat{B}'\tilde{x}(t)\| \leq \frac{1}{2}$ for $t \geq T_0$ and hence

$$\dot{\tilde{x}} = (\hat{A} - \hat{B}\hat{B}')\tilde{x}$$

and since this system matrix is Hurwitz stable, we have $\tilde{x}(t) \rightarrow 0$ as $t \rightarrow \infty$. For $t > T_0$, we have that

$$\dot{\tilde{x}} = A\bar{x} + B\sigma(-B'P\bar{x}) + \bar{\mathcal{I}}\tilde{x}$$

where $\bar{\mathcal{I}} = \mathcal{I} - \bar{B}\hat{B}'$. Define $V_2 = \bar{x}'P\bar{x}$ and a set

$$\mathcal{K} = \left\{ \bar{x} \mid V_2(\bar{x}) \leq \frac{\delta^2}{\text{trace}(B'P(1)B)} \right\}.$$

It can be easily seen from (16) that for $\bar{x} \in \mathcal{K}$, $\varepsilon_a(\bar{x}) = 1$. Next, consider the derivative of V_2 ,

$$\begin{aligned} \dot{V}_2 = & -\varepsilon V_2 - \tilde{x}'PBB'P\tilde{x} + 2\tilde{x}'P\bar{\mathcal{I}}\tilde{x} + \bar{x}'\frac{dP}{dt}\bar{x} \\ \leq & -\varepsilon V_2 + 2\tilde{x}'P\bar{\mathcal{I}}\tilde{x} + \bar{x}'\frac{dP}{dt}\bar{x} \\ \leq & -\varepsilon V_2 + 2\sqrt{V_2}\|P^{1/2}\bar{\mathcal{I}}\tilde{x}\| + \bar{x}'\frac{dP}{dt}\bar{x}. \end{aligned}$$

Note that \mathcal{I} , \bar{B} and hence $\bar{\mathcal{I}}$ are in the form of (12). Lemma 2 shows that there exists an M such that

$$\|P^{1/2}\bar{\mathcal{I}}\tilde{x}\| = \sqrt{\tilde{x}'\bar{\mathcal{I}}'P\bar{\mathcal{I}}\tilde{x}} \leq \varepsilon\sqrt{M}\|\tilde{x}\|.$$

We use here that Lemma 2 holds for any matrix of the form (12) so it also holds for \bar{E}_1 replaced by $\bar{\mathcal{I}}$. Hence

$$\begin{aligned} \dot{V}_2 \leq & -\varepsilon V_2 + 2\varepsilon\sqrt{M}\|\tilde{x}\|\sqrt{V_2} + \bar{x}'\frac{dP}{dt}\bar{x} \\ \leq & -\varepsilon\sqrt{V_2}\left[\sqrt{V_2} - 2\sqrt{M}\|\tilde{x}\|\right] + \bar{x}'\frac{dP}{dt}\bar{x}. \end{aligned}$$

Since $\tilde{x} \rightarrow 0$, there exists a $T_1 > T_0$ such that for $t \geq T_1$,

$$\|\tilde{x}\| \leq \frac{\delta}{4\sqrt{M}\sqrt{\text{trace}(B'P(1)B)}}.$$

Therefore, for $t \geq T_1$ and $\bar{x} \notin \mathcal{K}$ we have

$$\sqrt{V_2} - 2\sqrt{M}\|\tilde{x}\| \geq \frac{\sqrt{V_2}}{2}$$

and thus

$$\dot{V}_2 \leq -\frac{\varepsilon}{2}V_2 + \bar{x}'\frac{dP}{dt}\bar{x}.$$

Since \dot{V}_2 cannot have the same sign as $\bar{x}'\frac{dP}{dt}\bar{x}$ (see [1]), we conclude that $\dot{V}_2 < 0$ for $\bar{x} \notin \mathcal{K}$ and $t > T_1$. This implies that \bar{x} will enter \mathcal{K} within finite time, say $T_2 > T_1$, and remain in \mathcal{K} thereafter. For $t > T_2$ and $\bar{x} \in \mathcal{K}$, we have

$\varepsilon_a(\bar{x}) = 1$ and $\|\hat{B}'\bar{x}\| \leq \frac{1}{2}$. All saturations are inactive and the system becomes

$$\begin{cases} \dot{\bar{x}} = (A - BB'P(1))\bar{x} + \bar{\mathcal{L}}\bar{x}, \\ \dot{\hat{x}} = (\hat{A} - \hat{B}\hat{B}')\hat{x}. \end{cases}$$

The global asymptotic stability follows from the properties that $\hat{A} - \hat{B}\hat{B}'$ and $A - BB'P(1)$ are Hurwitz stable. We proceed to show the boundedness of trajectories in presence of d_1 and d_2 . Define

$$R = e^{\hat{A}t} \quad \text{and} \quad y = R\bar{x}.$$

Note that since $\hat{A} + \hat{A}' = 0$, R defines a rotation matrix. Moreover, we have that $\dot{R} = -R\hat{A}$. We obtain that

$$\dot{y} = R\hat{B}\sigma(-\hat{B}'R'y + v) - R\hat{B}\sigma(v) + R\hat{E}_2d_2 + R\hat{E}_3d_3$$

with $y(0) = \bar{x}_0$ where $v = -B'P\bar{x}$. Let \bar{y} satisfy

$$\dot{\bar{y}} = R\hat{E}_2d_2, \quad \bar{y}(0) = \bar{x}_0.$$

Since $d_2 \in \Omega_\infty$, we find that $\bar{y} \in \mathcal{L}_\infty$. Define $\tilde{y} = y - \bar{y}$. Then

$$\dot{\tilde{y}} = R\hat{B}\sigma(-\hat{B}'R'\tilde{y} - \hat{B}'R'\bar{y} + v) - R\hat{B}\sigma(v) + R\hat{E}_3d_3$$

with $\tilde{y}(0) = 0$. Again define $z = R'\tilde{y}$. We get

$$\dot{z} = \hat{A}z + \hat{B}\sigma(-\hat{B}'z - \hat{B}'R'\bar{y} + v) - \hat{B}\sigma(v) + \hat{E}_3d_3, \quad z(0) = 0,$$

Consider an auxiliary system

$$\dot{w} = (\hat{A} + \hat{B}\hat{F})w + \hat{E}_3d_3, \quad w(0) = 0,$$

where \hat{F} is such that $\hat{A} + \hat{B}\hat{F}$ is Hurwitz stable. For a selected \hat{F} , let δ_1 be sufficiently small such that $\|d_3\|_{\mathcal{L}_\infty} \leq \delta_1$ implies that $\|Fw\|_{\mathcal{L}_\infty} \leq 1/4$.

Let $\xi = z - w$. We have that

$$\dot{\xi} = \hat{A}\xi + \hat{B}\sigma(-\hat{B}'\xi + u) - \hat{B}\sigma(v) - \hat{B}\hat{F}w, \quad \xi(0) = 0,$$

where $u = -\hat{B}'w - \hat{B}'R'\bar{y} + v$. Since $u \in \mathcal{L}_\infty$ and $\|\sigma(v) + \hat{F}w\|_{\mathcal{L}_\infty} \leq 1/4 + 1/4 = 1/2$, it follows from Lemma 3 in the appendix that $\xi \in \mathcal{L}_\infty$. This implies that $\tilde{x} \in \mathcal{L}_\infty$.

Consider the dynamics of \bar{x}

$$\dot{\bar{x}} = A\bar{x} + B\sigma(-B'P\bar{x}) + \bar{B}\zeta + \bar{E}_1d_1 + \mathcal{I}\bar{x}$$

where $\zeta = \sigma(-\hat{B}'\bar{x} - B'P\bar{x}) - \sigma(-B'P\bar{x})$. Since $\sigma(\cdot)$ is globally Lipschitz with Lipschitz constant 1, we have that $\|\zeta\| \leq \|\hat{B}'\bar{x}\|$ and thus $\zeta \in \mathcal{L}_\infty$.

By differentiating $V_2 = \bar{x}'P\bar{x}$, we obtain

$$\begin{aligned} \dot{V}_2 &\leq -\varepsilon V_2 + 2x'P\bar{E}_1d_1 + 2x'P\mathcal{I}\bar{x} + 2\bar{x}'P\bar{B}\zeta + \bar{x}'\frac{dP}{dt}\bar{x} \\ &\leq -\varepsilon V_2 + 2\sqrt{V_2}\|P^{1/2}\bar{E}_1d_1\| + 2\sqrt{V_2}\|P^{1/2}\bar{B}\zeta\| \\ &\quad + 2\sqrt{V_2}\|P^{1/2}\mathcal{I}\bar{x}\| + \bar{x}'\frac{dP}{dt}\bar{x}. \end{aligned}$$

We have already shown in Lemma 2 that there exist M_1 , M_2 and M_3 such that

$$\begin{aligned} \|P^{1/2}\bar{E}_1d_1\| &\leq \varepsilon\sqrt{M_1}\|d_1\|, \quad \|P^{1/2}\bar{B}\zeta\| \leq \varepsilon\sqrt{M_2}\|\zeta\| \\ \text{and } \|P^{1/2}\mathcal{I}\bar{x}\| &\leq \varepsilon\sqrt{M_3}\|\bar{x}\|. \end{aligned}$$

We obtain,

$$\begin{aligned} \dot{V}_2 &\leq -\varepsilon\sqrt{V_2}\left[\sqrt{V_2} - 2\sqrt{M_1}\|d_1\|_\infty - 2\sqrt{M_3}\|\bar{x}\|_\infty \right. \\ &\quad \left. - 2\sqrt{M_2}\|\zeta\|_\infty\right] + \bar{x}'\frac{dP}{dt}\bar{x}. \end{aligned}$$

If $\sqrt{V_2} \geq 2\sqrt{M_1}\|d_1\|_\infty + 2\sqrt{M_3}\|\bar{x}\|_\infty + 2\sqrt{M_2}\|\zeta\|_\infty$, we have

$$\dot{V}_2 \leq \bar{x}'\frac{dP}{dt}\bar{x}.$$

Since \dot{V}_2 and $\bar{x}'\frac{dP}{dt}\bar{x}$ can not have the same sign, we find that $\dot{V}_2 \leq 0$ for

$$\bar{x} \in \left\{ \bar{x} \mid \sqrt{V_2} \geq 2\sqrt{M_1}\|d_1\|_\infty + 2\sqrt{M_3}\|\bar{x}\|_\infty + 2\sqrt{M_2}\|\zeta\|_\infty \right\},$$

which, from the property (4) of scheduling (16), implies that $\bar{x} \in \mathcal{L}_\infty$ and hence $x \in \mathcal{L}_\infty$. ■

APPENDIX

The following Lemma is adapted from [2].

Lemma 3: Assume that (A, B) is controllable and $A + A' = 0$. There exists δ_0 such that the system

$$\dot{x} = Ax - B\sigma(B'x + u) - v, \quad x(0) = 0$$

satisfies that $x \in \mathcal{L}_\infty$ for $u \in \mathcal{L}_\infty$ and $v \in \mathcal{L}_\infty(\delta_0)$.

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