Stabilization of Multiple-Input Multiple-Output Linear Systems with Saturated Outputs

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***Abstract*—We consider linear time-invariant multiple-input multiple-output systems that are controllable and observable, where each output component is saturated. We demonstrate by constructive design that such systems can be globally asymptotically stabilized by output feedback without further restrictions. This result is an extension of a previous result by Kreisselmeier for single-input single-output systems. The control strategy consists of driving the components of the output vector out of saturation one by one, to identify the state of the system. Deadbeat control is then applied to drive the state to the origin.**

I. INTRODUCTION

Saturations are ubiquitous in physical control systems, and occur both in actuators, states, and outputs. In this note we focus on linear time-invariant multiple-input multiple-output (MIMO) systems with saturated outputs. An output saturation typically occurs when a measured quantity exceeds the range of the sensor used to measure it. It can also occur as a result of a nonlinear measurement equation. An example of the latter can be found in the automotive industry, where the measured lateral acceleration of a car can be used to estimate its sideslip angle [1]. The response of the lateral acceleration to changes in the sideslip angle is approximately linear for small sideslip angles, but a saturation occurs for large sideslip angles.

Several results in the literature deal with the issue of output saturations. Kreisselmeier demonstrated in [2] that it is possible to design a control law for any linear time-invariant single-input single-output (SISO) system with a saturated output to make it globally asymptotically stable, provided the linear system is controllable and observable. It is not obvious that this should be possible, because globally stabilizable and observable systems may not be globally stabilizable by output feedback, as demonstrated in [3]. Observability of systems with saturated outputs was studied in detail in [4].

In [5], Lin and Hu presented a design that applies to stabilizable and detectable SISO systems with all the invariant zeros located in the closed left-half plane. The design in [5] is semiglobal, but it is based on a linear control law, unlike the discontinuous control law from [2]. As pointed out by the authors, the approach in [5] cannot easily be extended to MIMO systems. In [6], the result in [5] was extended to handle tracking of signals produced by marginally stable exosystems.

In [7], Kaliora and Astolfi presented an approach for global stabilization of linear systems with output saturations, under the conditions that the linear system is controllable and observable, and that the open-loop system is stable. The design in [7] is formulated for SISO systems, but it is also applicable to MIMO systems, as remarked by the authors. Recent results on anti-windup strategies for systems with output saturations (see, e.g., [8]), as well as an $\mathscr{H}_\infty$-based approach [9] for systems with output nonlinearities, also deal with MIMO systems. However, these methods can in general provide neither global nor semiglobal stabilization, unless the open-loop system is already asymptotically stable.

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To the best of our knowledge, no previous results address stabilization of general controllable and observable linear MIMO systems with output saturations, either globally or semiglobally. The purpose of this note is to demonstrate by constructive design that such stabilization is possible without further restrictions, by extending the result of Kreisselmeier for SISO systems. In Kreisselmeier's design, the output is first brought out of saturation using a control strategy that relies only on the sign of the output. When the output comes out of saturation, the state of the system is identified exactly by using a deadbeat observer. This is possible, even if the output is out of saturation only for a brief interval, because the system behaves like a linear observable system during that time. Once the state of the system has been identified, it is brought to the origin in finite time by using a deadbeat control strategy. At first glance, the MIMO case looks considerably more complicated. The analogous strategy would be to drive the output of the linear system into the hyperrectangle where every output component is unsaturated. It is exceedingly difficult to do so, however, because one needs to coordinate several output components to make them simultaneously unsaturated, based only on their signs.

The central point in this note is that it is unnecessary to make the output components simultaneously unsaturated. Instead it is sufficient to bring each component out of saturation at least once, even if some or all of the other components are saturated when this happens. Our control strategy is therefore to drive the components out of saturation one by one. The data gathered from each component when it was unsaturated is then pieced together to identify the state of the system. Finally, the state is brought to the origin by deadbeat control.

We emphasize that the focus of this note is not on issues of performance; rather, the goal is to prove that global asymptotic stabilization by output feedback is possible for the class of systems under considerations, and to illustrate the principle that information from multiple output components that come out of saturation at different points in time can be combined to identify the state of the system. Nevertheless, we present a numerical simulation example that illustrates the workings of the control law, and we discuss various numerical issues related to implementation of the control law.

II. PROBLEM FORMULATION

We consider a linear time-invariant system with saturated outputs:

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(t) \in \mathbb{R}^n, \quad u(t) \in \mathbb{R}^m, \tag{1a}$$

$$y(t) = \mathrm{sat}(Cx(t)), \qquad y(t) \in \mathbb{R}^p, \tag{1b}$$

where $\mathrm{sat}(\cdot)$ represents a standard component-wise saturation. That is,

$$y(t) = \begin{bmatrix} \mathrm{sat}(C_1 x(t)) \\ \vdots \\ \mathrm{sat}(C_p x(t)) \end{bmatrix},$$

where $\mathrm{sat}(C_i x(t)) = \mathrm{sign}(C_i x(t)) \min\{1, |C_i x(t)|\}$ and C_i, $i = 1, \dots, p$, are the rows of C.

Assumption 1: The pair (A, B) is controllable, and the pair (C, A) is observable.

We assume that the system is initialized at time $t = 0$. We seek to render to the origin of the system (1) globally asymptotically stable by output feedback.

III. CONTROL

For the purpose of control, we divide the time $t > 0$ into intervals $(kT, kT + T]$, where $k = 0, 1, 2, \dots$, and $T > 0$ is a constant chosen by the designer. On each interval, the control is given as in [2] by

$$u(kT+\tau) = -B^{\mathsf{T}} \mathrm{e}^{-A^{\mathsf{T}}\tau} U_k, \quad \tau \in (0,T], \tag{2}$$

where U_k is constant on the interval $(kT, kT+T]$. We define U_k as follows:

$$U_k = \begin{cases} \bar{B}^{-1}\alpha^k(\alpha h_{j_k} - \mathrm{e}^{AT}h_{j_k})y_{j_k}(kT), & \prod_{i=1}^{p}\sigma_i(kT) = 0, \\ \bar{B}^{-1}\mathrm{e}^{AT}(\mathrm{e}^{AkT}D^{-1}(kT)\xi(kT) + \mu(kT)), & \prod_{i=1}^{p}\sigma_i(kT) > 0, \end{cases} \tag{3}$$

where j_k is defined as the smallest $i \in 1,\dots,p$ such that $\sigma_i(kT) = 0$. The vectors $h_i \in \mathbb{R}^n$, $i = 1,\dots,p$, are chosen by the designer in such a way that $C_i h_i > 0$. The scalar α is defined as $\alpha = \rho\|\exp(AT)\|$, where $\rho > 1$ is a number chosen by the designer. The constant matrix $\bar{B} \in \mathbb{R}^{n\times n}$ is defined as

$$\bar{B} = \int_0^T \mathrm{e}^{A(T-\tau)}BB^{\mathsf{T}}\mathrm{e}^{-A^{\mathsf{T}}\tau}\,\mathrm{d}\tau. \tag{4}$$

We note that $\bar{B}$ is invertible, due to controllability of the pair (A,B). The quantities $\sigma_i(t) \in \mathbb{R}$, $i = 1,\dots,p$, $\xi(t) \in \mathbb{R}^n$, $\mu(t) \in \mathbb{R}^n$, and $D(t) \in \mathbb{R}^{n\times n}$ are given by the following expressions:

$$\sigma_i(t) = \int_0^t (1 - |y_i(\tau)|)\,\mathrm{d}\tau, \quad i = 1,\dots,p, \tag{5a}$$

$$\xi(t) = \sum_{i=1}^{p}\int_0^t (1-|y_i(\tau)|)\mathrm{e}^{A^{\mathsf{T}}\tau}C_i^{\mathsf{T}}\left(y_i(\tau) - C_i\mu(\tau)\right)\mathrm{d}\tau, \tag{5b}$$

$$\mu(t) = \int_0^t \mathrm{e}^{A(t-\tau)}Bu(\tau)\,\mathrm{d}\tau, \tag{5c}$$

$$D(t) = \sum_{i=1}^{p}\int_0^t (1-|y_i(\tau)|)\mathrm{e}^{A^{\mathsf{T}}\tau}C_i^{\mathsf{T}}C_i\mathrm{e}^{A\tau}\,\mathrm{d}\tau. \tag{5d}$$

In the following section, we provide a simple explanation of how the control algorithm works. The details of this explanation will become clear from the proof of Theorem 1 in Section III-B.

A. *Explanation of the Control Law*

The control strategy is based on dividing the time $t > 0$ into intervals $(kT, kT+T]$. At the beginning of each interval, the control to be applied over the interval is determined by calculating the vector U_k according to (3). The scalar functions $\sigma_i(t)$, $i = 1,\dots,p$, play a crucial role in this process. Each signal $\sigma_i(t)$ is an indicator of whether the corresponding output component $y_i(t)$ has been out of saturation at any time since initialization at time $t = 0$. If $y_i(t)$ has been saturated the whole time, then $\sigma_i(t) = 0$; if $y_i(t)$ has been out of saturation at any point since initialization, then $\sigma_i(t) > 0$.

1) Driving the Outputs Out of Saturation: The initial task of the controller is to ensure that each output component $y_i(t)$ comes out of saturation at least once. This task is accomplished sequentially, by first ensuring that $y_1(t)$ comes out of saturation, then ensuring that $y_2(t)$ comes out of saturation, and so on. At the start of an interval $(kT, kT+T]$, the value j_k is set to the smallest i such that $\sigma_i(kT) = 0$, meaning that $y_i(t)$ has not yet been out of saturation. Assuming such an i exists, U_k is set to $U_k = \bar{B}^{-1}\alpha^k(\alpha h_{j_k} - \mathrm{e}^{AT}h_{j_k})y_{j_k}(kT)$, according to (3). This choice of U_k is the same as the one used in [2] to drive the output of a SISO system out of saturation. The strategy behind this choice is to drive the output in the direction of the origin, based only on the sign of the output. For each interval that passes, the control amplitude grows larger, due to the factor α^k. The increasing control amplitude is needed in order to catch up with any instabilities in the system that may be driving the output away from the origin.

2) Deadbeat Control: Eventually, every output component will have been out of saturation at least once, and thus there will be an integer $\hat{k}$ such that at time $\hat{k}T$, we have $\sigma_i(\hat{k}T) > 0$ for all $i \in 1,\dots,p$. This can also be expressed as $\prod_{i=1}^{p}\sigma_i(\hat{k}T) > 0$. At this point, $U_{\hat{k}}$ is chosen as $U_{\hat{k}} = \bar{B}^{-1}\mathrm{e}^{AT}(\mathrm{e}^{A\hat{k}T}D^{-1}(\hat{k}T)\xi(\hat{k}T) + \mu(\hat{k}T))$, according to (3). To justify this choice, we first note that the initial condition of the system can be related to the quantities in (5) by the expression $\xi(\hat{k}T) = D(\hat{k}T)x(0)$ (this relationship will become clear from the proof of Theorem 1 below). Since every component of the output has been out of saturation at least once, $D(\hat{k}T)$ is a nonsingular matrix, and hence $x(0)$ can be calculated as $x(0) = D^{-1}(\hat{k}T)\xi(\hat{k}T)$. Once $x(0)$ is known, $x(\hat{k}T)$ can also be calculated from the standard variation-of-constants formula [10] as $x(\hat{k}T) = \mathrm{e}^{A\hat{k}T}x(0) + \int_0^{\hat{k}T}\mathrm{e}^{A(\hat{k}T-\tau)}Bu(\tau)\,\mathrm{d}\tau = \mathrm{e}^{A\hat{k}T}D^{-1}(\hat{k}T)\xi(\hat{k}T) + \mu(\hat{k}T)$. We have therefore identified the state of the system at time $\hat{k}T$ precisely. We can now rewrite $U_{\hat{k}}$ as $U_{\hat{k}} = \bar{B}^{-1}\mathrm{e}^{AT}x(\hat{k}T)$, which yields the same deadbeat control as in [2] and ensures that $x(\hat{k}T+T) = 0$. The difference between this note and [2] lies in how the information from different output components, which become unsaturated at different times, is pieced together to allow deadbeat observation of the state of the system.

Remark 1: We remark that α in this note is chosen somewhat differently from [2], where α was defined as $\alpha = \exp(2\|A\|T)$. The point of redefining α is to ensure that the factor α^k does not grow much faster than what is necessary to drive the output components out of saturation. The definition used here also ensures that $\alpha > \|\exp(AT)\|$, even when $A = 0$. Technically, an additional condition of this kind is also needed in [2], to handle the particular case where the system consists of a single integrator.

B. *Stability*

We now state the stability results of the closed-loop system in a formal manner.

Theorem 1: The origin of (1) with the control (2)–(5) is globally asymptotically stable.

Proof: Given t_1 and t_2 such that $t_2 \geq t_1 \geq 0$, we know from the standard variation-of-constants formula [10] that

$$x(t_2) = \mathrm{e}^{A(t_2-t_1)}x(t_1) + \int_0^{t_2-t_1}\mathrm{e}^{A(t_2-t_1-\tau)}Bu(t_1+\tau)\,\mathrm{d}\tau. \tag{6}$$

Applying this formula over an interval by setting $t_1 = kT$ and $t_2 = kT+T$, and inserting the expression (2) for the control law, it is easily confirmed that we obtain, as in [2],

$$x(kT+T) = \mathrm{e}^{AT}x(kT) - \bar{B}U_k. \tag{7}$$

We shall use the expression (7) in the remainder of the proof, which is divided into two parts, similar to [2]. First, we show that the origin is globally attractive, and then we show that it is a Lyapunov stable equilibrium point.

The origin is globally attractive: We start by showing that there is an integer $\hat{k}$ such that each output component has been out of saturation on $[0, \hat{k}T]$. For the sake of establishing a contradiction, suppose that this is not the case. Then there is a set of output components that remain in saturation for all time. Let l denote the smallest integer such that $y_l(t)$ remains in saturation for all time. Then there is an integer $\bar{k}$ such that for all $k \geq \bar{k}$, $\sigma_i(kT) > 0$ for all $i = 1,\dots,l-1$. It follows that $j_k = l$ for all $k \geq \bar{k}$. We now establish the contradiction by showing that $y_l(t)$ will eventually become unsaturated. This is established in the same way as in the proof from [2].

Since $j_k = l$ for all $k \geq \bar{k}$, we have from (3) that for all $k \geq \bar{k}$, $U_k = \bar{B}^{-1}\alpha^k(\alpha h_l - \mathrm{e}^{AT}h_l)y_l(kT)$. Inserting this expression for U_k into

(7), we see that for all $k \geq \bar{k}$, $x(kT+T) = \mathrm{e}^{AT}x(kT) - \alpha^k(\alpha h_l - \mathrm{e}^{AT}h_l)y_l(kT)$, which can be reformulated as

$$x(kT+T) = \mathrm{e}^{AT}(x(kT) + \alpha^k h_l y_l(kT)) - \alpha^{k+1}h_l y_l(kT). \quad (8)$$

Since $y_l(t)$ is assumed to be saturated for all time, we can replace $y_l(kT)$ in (8) by $y_l(\bar{k}T)$. We can furthermore calculate the solution recursively as

$$x(kT+T) = \mathrm{e}^{A\tilde{k}T}(x(\bar{k}T) + \alpha^{\bar{k}}h_l y_l(\bar{k}T)) - \alpha^{k+1}h_l y_l(\bar{k}T),$$

where $\tilde{k} = k - \bar{k} + 1$. Premultiplying this expression by $y_l(\bar{k}T)C_l$ and making the substitution $k+1 = \tilde{k} + \bar{k}$ gives

$$y_l(\bar{k}T)y_l(kT+T) = C_l\mathrm{e}^{A\tilde{k}T}(x(\bar{k}T)y_l(\bar{k}T) + \alpha^{\bar{k}}h_l) - \alpha^{\tilde{k}+\bar{k}}C_l h_l.$$

We now upper-bound the first term on the right-hand side, using the expression $\alpha^{\tilde{k}} = \rho^{\tilde{k}}\|\exp(AT)\|^{\tilde{k}}$, which yields

$$y_l(\bar{k}T)y_l(kT+T) \leq \|\mathrm{e}^{AT}\|^{\tilde{k}}\left(\|C_l\|(\|x(\bar{k}T)\| + \alpha^{\bar{k}}\|h_l\|) - \rho^{\tilde{k}}\alpha^{\bar{k}}C_l h_l\right).$$

Considering the expression inside the parenthesis, we see that the first term remains constant as k, and therefore $\tilde{k}$, grows larger, whereas the second term becomes more and more negative (recall that $C_l h_l > 0$). Thus, we eventually have $y_l(\bar{k}T)y_l(kT+T) < 0$, which implies that the sign of $y_l(kT+T)$ is different from the sign of $y_l(\bar{k}T)$. This contradicts the assumption that $y_l(t)$ remains saturated for all time, and we have therefore proven that there exists an integer $\hat{k}$ such that all the output components have been out of saturation at least once on $[0, \hat{k}T]$.

It follows from the above discussion that $\prod_{i=1}^{p}\sigma_i(\hat{k}T) > 0$. Thus, we have from (3) that

$$U_{\hat{k}} = \bar{B}^{-1}\mathrm{e}^{AT}(\mathrm{e}^{A\hat{k}T}D^{-1}(\hat{k}T)\xi(\hat{k}T) + \mu(\hat{k}T)).$$

As discussed in Section III-A2, the idea behind this choice is to make $U_{\hat{k}}$ satisfy the expression $U_{\hat{k}} = \bar{B}^{-1}\mathrm{e}^{AT}x(\hat{k}T)$. This would result in deadbeat control as in [2], which is easily seen by inserting the expression $U_{\hat{k}} = \bar{B}^{-1}\mathrm{e}^{AT}x(\hat{k}T)$ into (7), to obtain $x(\hat{k}T+T) = \mathrm{e}^{AT}x(\hat{k}T) - \mathrm{e}^{AT}x(\hat{k}T) = 0$. We now need to show that $\mathrm{e}^{A\hat{k}T}D^{-1}(\hat{k}T)\xi(\hat{k}T) + \mu(\hat{k}T) = x(\hat{k}T)$.

Using (6) with $t_1 = 0$ and $t_2 = t$, it is easily confirmed that we have $x(t) = \mathrm{e}^{At}x(0) + \mu(t)$. Premultiplying this expression by $(1 - |y_i(t)|)\mathrm{e}^{A^\mathsf{T}t}C_i^\mathsf{T}C_i$ for any $i \in 1,\ldots,p$ and rearranging yields

$$(1-|y_i(t)|)\mathrm{e}^{A^\mathsf{T}t}C_i^\mathsf{T}(y_i(t) - C_i\mu(t)) = (1-|y_i(t)|)\mathrm{e}^{A^\mathsf{T}t}C_i^\mathsf{T}C_i\mathrm{e}^{At}x(0),$$

where we have used the fact that $C_i x_i(t) = y_i(t)$ whenever $(1 - |y_i(t)|) \neq 0$. If we now integrate this expression from 0 to $\hat{k}T$ and take the sum over i from 1 to p, we obtain the expression $\xi(\hat{k}T) = D(\hat{k}T)x(0)$. Assuming for the moment that $D(\hat{k}T)$ is nonsingular, we can calculate the initial condition as $x(0) = D^{-1}(\hat{k}T)\xi(\hat{k}T)$. Again using (6) with $t_1 = 0$ and $t_2 = \hat{k}T$, we obtain the desired expression $\mathrm{e}^{A\hat{k}T}D^{-1}(\hat{k}T)\xi(\hat{k}T) + \mu(\hat{k}T) = x(\hat{k}T)$.

We still have to show that $D(\hat{k}T)$ is nonsingular. We can write $D(\hat{k}T) = \sum_{i=1}^{p}D_i(\hat{k}T)$, where

$$D_i(\hat{k}T) = \int_0^{\hat{k}T}(1-|y_i(\tau)|)\mathrm{e}^{A^\mathsf{T}\tau}C_i^\mathsf{T}C_i\mathrm{e}^{A\tau}\,\mathrm{d}\tau. \quad (9)$$

Each matrix $D_i(\hat{k}T)$ is positive semidefinite, because the integrand in (9) is positive semidefinite. It follows that $D(\hat{k}T)$ is also positive semidefinite. We shall prove that $D(\hat{k}T)$ is in fact positive *definite*, by showing that $z^\mathsf{T}D(\hat{k}T)z > 0$ for each $z \neq 0$.

Since each component of the output has been out of saturation on the interval $[0, \hat{k}T]$ and the solutions are continuous, there exists a number $m > 0$ and a set of intervals $(t_i, t_i+\varepsilon) \subset (0, \hat{k}T)$, $i = 1,\ldots,p$, of length $\varepsilon > 0$ such that for all $t \in (t_i, t_i+\varepsilon)$, $1 - |y_i(t)| > m$. It therefore follows that

$$\begin{aligned} z^\mathsf{T}D(\hat{k}T)z &= \sum_{i=1}^{p} z^\mathsf{T}D_i(\hat{k}T)z \\ &> m\sum_{i=1}^{p}\int_{t_i}^{t_i+\varepsilon} z^\mathsf{T}\mathrm{e}^{A^\mathsf{T}\tau}C_i^\mathsf{T}C_i\mathrm{e}^{A\tau}z\,\mathrm{d}\tau. \end{aligned} \quad (10)$$

Suppose for the sake of establishing a contradiction that there exists a $z \neq 0$ such that the right-hand side of (10) is zero. This implies that for each $i \in 1,\ldots,p$ and for all $t \in (t_i, t_i+\varepsilon)$, $z^\mathsf{T}\mathrm{e}^{A^\mathsf{T}t}C_i^\mathsf{T}C_i\mathrm{e}^{At}z = 0$. This furthermore implies that for each $i = 1,\ldots,p$ and for all $t \in (t_i, t_i+\varepsilon)$, $C_i\mathrm{e}^{At}z = 0$, which means that for all $t \in (t_i, t_i+\varepsilon)$, $\mathrm{e}^{At}z$ must belong to the unobservable subspace of the pair (C_i, A). Since the unobservable subspace is A-invariant (see [10, Ch. 21]), the vector z must also belong to the unobservable subspace with respect to the pair (C_i, A). However, if the vector z belongs to unobservable subspace of (C_i, A) for each $i \in 1,\ldots,p$, then it belongs to the unobservable subspace of (C, A), which consists only of the origin. This shows that the right-hand side of (10) cannot be zero unless $z = 0$, and it follows that $D(\hat{k}T)$ is positive definite (and therefore nonsingular).

The above argument proves that $U_{\hat{k}}$ satisfies $U_{\hat{k}} = \bar{B}^{-1}\mathrm{e}^{AT}x(\hat{k}T)$, so that deadbeat control is applied over the interval $(\hat{k}T, \hat{k}T+T]$. This yields $x(\hat{k}T+T) = 0$. The same argument can be used to show that $U_{\hat{k}+1} = \bar{B}^{-1}\mathrm{e}^{AT}x(\hat{k}T+T) = 0$, and thus the control $u(t) = 0$ is applied over the interval $(\hat{k}T+T, \hat{k}T+2T]$. It follows that $x(t)$ remains at the origin on this interval, and by induction, on every subsequent interval.

The origin is a Lyapunov stable equilibrium point: Lyapunov stability is established by the same approach as in [2]. Suppose that $\|x(0)\|$ is sufficiently small such that all the output components are unsaturated at $t = 0$ (i.e., $|y_i(0)| < 1$, $i = 1,\ldots,p$). Clearly $\sigma_i(0) = 0$ for all $i \in 1,\ldots,p$. Hence, $j_0 = 1$, and U_0 is given by $U_0 = \bar{B}^{-1}(\alpha h_1 - \mathrm{e}^{AT}h_1)y_1(0)$, according to (3). It follows that $\|U_0\|$ is bounded by an expression proportional to $\|y_1(0)\|$, and therefore also by an expression proportional to $\|x(0)\|$. Using (6), it is now straightforward to show that there is a $\gamma_1 > 1$ such that for all $t \in (0, T]$, $\|x(t)\| \leq \gamma_1\|x(0)\|$. Since every output component is out of saturation for at least part of the first interval, we have $\sigma_i(T) > 0$ for all $i \in 1,\ldots,p$. By the above discussion of global attractivity, we therefore have $U_1 = \bar{B}^{-1}\mathrm{e}^{AT}x(T)$, and it follows that $\|U_1\|$ is bounded by an expression proportional to $\|x(T)\|$. Using (6), it is now straightforward to show that there is a $\gamma_2 > 1$ such that for all $t \in (T, 2T]$, $\|x(t)\| \leq \gamma_2\|x(T)\|$. For all $t > 2T$, we have $x(t) = 0$. Combining the above inequalities, we therefore find that for sufficiently small $\|x(0)\|$, $\|x(t)\| \leq \gamma_1\gamma_2\|x(0)\|$ for all $t \geq 0$. This shows that the origin is a Lyapunov stable equilibrium point. ■

IV. Numerical Issues and Tuning

As mentioned above, the input applied to the system to drive the outputs out of saturation grows larger with each time interval, in order to catch up with any instabilities in the system. As a consequence of this, the state of the system may grow large, and an output component may come out of saturation only briefly before again becoming saturated. The calculation of the state of the system may therefore be poorly conditioned and sensitive to measurement noise, disturbances, and model inaccuracies.

Applying a growing input is only necessary in order to handle instabilities; for systems that are open-loop stable (for example, as in [7]), there is no danger of the state escaping, and so a small input may be applied over a long period of time to bring all the outputs out of saturation. An alternative to applying a growing input is to adjust the controller parameters according to the size of an admissible set of initial conditions, leading to semiglobal results, as in [5].

We follow the approach of [2] in applying deadbeat control to bring the state to the origin. However, deadbeat control is rarely used in practical implementations, in particular, due to robustness problems. There is no theoretical requirement that deadbeat control be used in the present case; once the state of the system has been identified, it is in principle known for all future time, and thus any state-feedback controller may be applied instead of the deadbeat controller.

Several of the quantities used in the controller can be adjusted by the designer, in particular, T, ρ, and h_i, $i = 1,\dots,p$. These quantities can be viewed as tuning parameters with associated tradeoffs. Choosing T small ensures that, once an output has been driven out of saturation, the controller quickly jumps to the next task, which may be to drive another output out of saturation or to apply deadbeat control. On the other hand, using a small T may cause the identification of the system state to be based on a smaller amount of data, thereby increasing sensitivity to uncertainties such as measurement noise. Choosing ρ low limits the growth rate of the input, but may increase the time spent on bringing the outputs out of saturation. The direction of the vector h_i affects the direction of the input applied to bring the output $y_i(t)$ out of saturation. Without further knowledge about the state, however, it is difficult to interpret how different choices affect the outcome. One option is therefore to use $h_i = C_i^{\mathsf{T}}$, possibly with a scaling to adjust the magnitude of the applied input.

We end this section by remarking that, although the integrals in (5) are always well-defined, some of them may grow unbounded as $t \to \infty$. The reason for this is that the controller needs to piece together data from different output components that become unsaturated at different times. The integrals in (5) are used to gather the necessary data from the time of initialization, without any form of forgetting. In a practical implementation it is obviously undesirable to have unbounded internal signals. However, the integrals in (5) are only needed up until the point when deadbeat control is applied. After deadbeat control has been applied, the state is at the origin and the outputs are all out of saturation. Thus, the issue of unbounded internal signals can easily be resolved by switching to a different controller (e.g., a linear control law) after deadbeat control has been applied. Alternatively, the controller algorithm in (2)–(5) may be reset with regular intervals from this point on.

V. Simulation Example

Consider the system

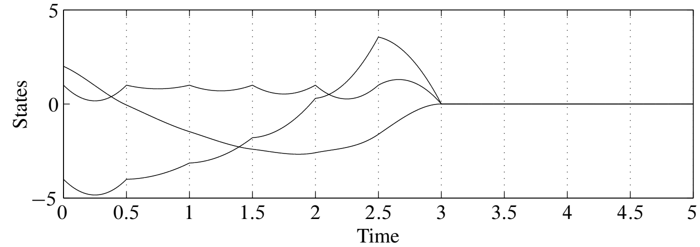
$$\dot{x}_1(t) = x_2(t) + x_3(t), \quad \dot{x}_2(t) = u_1(t), \quad \dot{x}_3(t) = u_2(t),$$
$$y_1(t) = x_1(t), \quad y_2(t) = x_1(t) + x_2(t).$$

This system has an eigenvalue with multiplicity one and an eigenvalue with multiplicity two, both at the origin; thus the system is open-loop unstable. We implement the controller algorithm using $T = 0.5$, $\rho = 1.1$, $h_1 = C_1^{\mathsf{T}}$, and $h_2 = C_2^{\mathsf{T}}$ and simulate with initial conditions $x_1(0) = 2$, $x_2(0) = -4$, and $x_3(0) = 1$. The results can be seen in Figure 1, where the intervals are marked by vertical dashed lines.

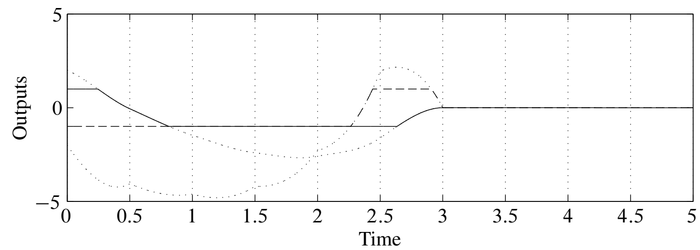
The controller begins by driving $y_1(t)$ out of saturation, and achieves this in the first time interval. The controller then proceeds with driving $y_2(t)$ out of saturation, which is achieved in the fifth time interval. At this point $y_1(t)$ is again saturated. In the sixth time interval deadbeat control is applied, so that for $t \geq 6T$, the state is at rest at the origin.

VI. Concluding Remarks

We have demonstrated that the origin of a linear time-invariant MIMO system with saturated outputs can be globally asymptotically



(a) State



(b) Outputs (solid: y_1, dashed: y_2). Unsaturated versions of the outputs are shown with dotted lines.

Fig. 1. Simulation results

stabilized, provided the linear system is controllable and observable. We have done so by extending the design presented in [2] for SISO systems. We note, however, that if the controller presented in this note is applied to a SISO system, it does not coincide precisely with the controller in [2]. This is because deadbeat identification of the state in [2] is done using data only from the previous interval.

The focus of this note is not on performance; indeed, it is unlikely that an unmodified version of the controller presented here would be suitable for practical implementation. Nevertheless, the results illustrate a general principle, namely, that one may drive each component of the output out of saturation separately to identify the state of the system, and thereafter control the state to the origin. Within this framework, both the approach used to drive the outputs out of saturation and the controller used to drive the state to the origin can be modified to improve performance.

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