High-Gain Observer Design for Domination of Nonlinear Perturbations: Transformation to a Canonical Form by Dynamic Output Shaping

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***Abstract*— We consider the problem of observer design for a class of systems described by an observable linear system perturbed by nonlinear, time-varying terms. In particular, we investigate conditions under which the effect of the nonlinear terms may be dominated by using sufficiently high gain. Our main tool is the transformation of the original system to a canonical form suitable for high-gain design, which is similar or equivalent to canonical forms used elsewhere in the high-gain literature. We demonstrate that linear, nonsingular transformations to this canonical form can easily be constructed using available tools, but that many systems do not allow for such a transformation. Our main contribution is a constructive algorithm that aims to rectify this problem by dynamically shaping the output.**

I. Introduction

In observer design it is common to encounter systems that are predominantly described by an observable linear time-invariant part, but that also include nonlinear and time-varying terms. Such systems can be described by

$$\dot{x} = Ax + \phi(t, x), \quad y = Cx, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $y \in \mathbb{R}^p$ is the output, and (C, A) is an observable pair. In some cases the nonlinearities may be exploited to enhance the stability properties of an observer; in other cases, the nonlinearities create an undesirable influence that must be treated as an uncertainty to be dominated. In the latter case, one often looks for a Lyapunov-type formulation that guarantees stability if the observer gains are chosen in a particular way. A typical result is that stability is ensured if some of the gains are chosen sufficiently high.

It is not always possible to dominate the effect of nonlinearities by increasing gains. *High-gain observer theory* is a theoretical framework that aims to classify the types of systems for which such domination is possible, and to specify how it may be achieved. Typically, one assumes that the nonlinearities are globally Lipschitz continuous (or at least locally Lipschitz continuous within some region of interest) with respect to the state, uniformly in time. Beyond that, the question of whether domination is possible depends on the structural relationship between the nonlinearities and the outputs.

The most basic case that allows for domination is a system consisting of a scalar nonlinearity separated from a scalar output by a chain of integrators. A more general case allows for an additional zero dynamics subsystem that is asymptotically stable and that affects the integrator chain at the same point as the nonlinearity. A fundamental property of such systems is that they can be written as

$$\dot{x} = Ax + E\psi(t, x), \quad y = Cx, \tag{2}$$

where y and $\psi(t, x)$ are scalar and the triple (C, A, E) is left-invertible and of minimum phase. In fact, the high-gain formulation used for such systems can be extended to general systems (with multiple outputs and multiple independent nonlinearities) on the form (2) for which (C, A, E) is left-invertible and of minimum phase. The high-gain observer design problem for such a triple is dual to the high-gain feedback design problem, for which much of the early high-gain theory was developed; for an overview, see [1] and references therein. High-gain observers were used early on in the context of *loop transfer recovery* [2], and later for nonlinear systems [3], [4]. For a recent review of high-gain observers in nonlinear feedback control, see [5].

A. High-Gain without Left-Invertibility or Minimum Phase

The conditions of left-invertibility and minimum phase are sensible when the goal is to suppress an uncertainty about which little or nothing is known. However, these conditions are often too stringent when the uncertainty is due to a nonlinearity whose dependency on the states of the system is known. This was demonstrated in [6], which treated single-output systems in the lower-triangular form $\dot{x}_i = x_{i+1} + \phi_i(x_1, \dots, x_i)$, $i = 1, \dots, n-1$, $\dot{x}_n = \phi_n(x_1, \dots, x_n)$, with $y = x_1$. Such systems can generally not be expressed in terms of a left-invertible triple, since the number of independent nonlinearities is greater than the number of outputs.

Generalizing the design from [6] to multiple-output systems turns out to be a complicated matter. Many contributions have been made on this topic, for example, [7]–[15]. These results are based on various canonical forms that generalize the chained, lower-triangular structure of the single-output case [6] to multiple chains. The most general form is probably described in [7], [12], where the chains are allowed to interact in a relatively complex manner. As pointed out in [15], however, applying the results from [7], [12] requires identifying a set of integers that can be difficult to find in practice.

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Two crucial questions that often receive little attention is when and how a given system can be transformed to a relevant canonical form. In some cases, for example [10], [11], [14], the existence of an appropriate coordinate change can be guaranteed if the system satisfies certain nonlinear observability conditions. However, these conditions are typically hard to confirm and provide little insight regarding how one might construct the coordinate change as a practical matter. A natural approach is to define new coordinates by taking repeated Lie derivatives of the output. In addition to the drawback of often producing highly complicated transformations, this approach is generally not successful when applied to multiple-output systems. This problem is demonstrated in [16], which proposes a procedure that consists of taking repeated Lie derivatives of the output and effectively discarding problematic output components. However, this procedure is likely to waste crucial output information, and it may therefore fail even for very simple, uniformly observable system, as illustrated by [16, Ex. 3].

B. Topics of this Paper

In this paper, our initial focus is on designing an observer for the system (1), on the standard form

$$\dot{\hat{x}} = A\hat{x} + \phi(t,\hat{x}) + K(y - C\hat{x}), \tag{3}$$

where K is a constant gain matrix. Defining the estimation error $\tilde{x} = x - \hat{x}$, this leads to the error dynamics

$$\dot{\tilde{x}} = (A - KC)\tilde{x} + \phi(t,x) - \phi(t,\hat{x}). \tag{4}$$

Our working assumption is that the term $\phi(t,x) - \phi(t,\hat{x})$ does not contribute toward stability, and that the gain K must therefore be designed to dominate its effect. Our procedure will be based on transformation to a canonical form that is similar to the canonical forms used in several of the papers [7]–[16]. When transforming systems to the canonical form, we are only interested in *linear* state and output transformations that can be constructed from A and C alone; that is, transformations that do not depend on the nonlinearity. We shall give precise conditions for when such transformations exist, and demonstrate how the transformations may be constructed using available tools and software.

As we shall see, it is often impossible to construct nonsingular transformations to bring the system to the desired canonical form. However, we shall demonstrate that it may still be possible to achieve the canonical form, by first shaping the outputs through the addition of stable output filters. The purpose of this dynamic shaping is to introduce a larger number of inherent integrations between the outputs and certain subspaces of the state space. This brings us to the main contribution of this paper, which is an algorithm that systematically shapes the output in order to make a system transformable to the canonical form.

II. Problem Formulation

We assume that $\phi(t,x)$ is globally Lipschitz continuous, uniformly in t, piecewise continuous in t, and continuously differentiable with respect to x.[1] We may therefore use the mean-value theorem to write

$$\phi(t,x) - \phi(t,\hat{x}) = \sum_{k=1}^{v} \mu_k(t,x,\hat{x}) W_k \tilde{x}. \tag{5}$$

In this expression, $\mu_k(t,x,\hat{x})$, $k = 1,\ldots,v$, are functions arising from partial derivatives of the nonlinearities, whose absolute values are uniformly bounded by constants $\mu_{k\,\max}$, due to the global Lipschitz assumption. It is our presumption that these functions are linearly independent, although our design does not depend on it. Furthermore, W_k, $k = 1,\ldots,v$, are known matrices that represent the structural dependency of the nonlinear vector function $\phi(t,x) = [\phi_1(t,x) \;\; \cdots \;\; \phi_n(t,x)]^\top$ on the states of the system. In particular, if $[\partial\phi_i/\partial x_j](t,x) = 0$, then element (i,j) of each W_k should be zero. As an example, consider a function $\phi(t,x) = [\phi_1(t,x_1,x_3) \;\; 0 \;\; 0]^\top$. We may write

$$\phi(t,x) - \phi(t,\hat{x}) = \mu_1(t,x,\hat{x}) \begin{bmatrix} 1&0&0\\0&0&0\\0&0&0 \end{bmatrix} \tilde{x} + \mu_2(t,x,\hat{x}) \begin{bmatrix} 0&0&1\\0&0&0\\0&0&0 \end{bmatrix} \tilde{x},$$

where $\mu_1(t,x,\hat{x})$ and $\mu_2(t,x,\hat{x})$ represent partial derivatives of $\phi_1(t,x_1,x_3)$ with respect to x_1 and x_3, respectively, taken at some points on the line between (x_1,x_3) and $(\hat{x}_1,\hat{x}_3)$.

We shall use this representation to reformulate our problem. Specifically, we seek to design an observer for the linear time-varying system

$$\dot{x} = Ax + \sum_{k=1}^{v} \omega_k(t) W_k x, \quad y = Cx. \tag{6}$$

We shall construct the gains of our observer without *a priori* information about the signals $\omega_k(t)$, except that they are bounded. In Section III-C, we show how the resulting design can be applied to the original nonlinear system (1).

III. Canonical Form

We now introduce a canonical form for (6), which is similar to the forms used in a number of previous articles. For example, the forms used in [9], [16], [14] are essentially nonlinear variations of the canonical form employed here.

To distinguish between the system (6) and a system in the canonical form, we use χ and γ to denote the state and output vectors of the system in the canonical form, and we use $\bar{A}$, $\bar{C}$, and $\bar{W}_k$, $k = 1,\ldots,v$ to denote the system matrices. Thus, the system equations are given by

$$\dot{\chi} = \bar{A}\chi + \sum_{k=1}^{v} \omega_k(t) \bar{W}_k \chi, \quad \gamma = \bar{C}\chi. \tag{7}$$

In the canonical form the state χ is partitioned as $\chi = [\chi_1^\top \;\; \cdots \;\; \chi_r^\top]^\top$, where each χ_i, $i = 1,\ldots,r$, is of dimension q_i, and the output γ is partitioned as $\gamma = [\gamma_1 \;\; \cdots \;\; \gamma_r]^\top$, where each γ_i, $i = 1,\ldots,r$, is scalar. Each

[1] As in other places in the literature, the global Lipschitz assumption can be relaxed to a local one, within some region of interest.

χ_i, $i = 1, \ldots, r$, represents its own subsystem, with system equations

$$\dot{\chi}_i = \bar{A}_{q_i}\chi_i + \bar{L}_i\gamma + \sum_{k=1}^{v}\sum_{j=1}^{r}\omega_k(t)\bar{W}_{kij}\chi_j, \quad \gamma_i = \bar{C}_{q_i}\chi_i,$$

where $\bar{A}_{q_i}$ and $\bar{C}_{q_i}$ are matrices with the special structure

$$\bar{A}_{q_i} = \begin{bmatrix} 0 & I_{q_i-1} \\ 0 & 0 \end{bmatrix}, \quad \bar{C}_{q_i} = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}.$$

This structure implies that the time-invariant part of each subsystem consists of an integrator chain of length q_i that terminates with a scalar output, plus a term $\bar{L}_i\gamma$ that depends only on the output of the system. In addition to the special structure of $\bar{A}_{q_i}$ and $\bar{C}_{q_i}$, we require each $\bar{W}_{kij}$, $k = 1, \ldots, v$, $i, j = 1, \ldots, r$, to be lower triangular; that is, element (c, r) of $\bar{W}_{kij}$ is zero whenever $c > r$ (note that $\bar{W}_{kij}$ is in general not a square matrix).

The matrices $\bar{A}$, $\bar{C}$, and $\bar{W}_k$, $k = 1, \ldots, v$, for the overall system are composed of $\bar{A}_{q_i}$, $\bar{L}_i$, $\bar{C}_{q_i}$, and $\bar{W}_{kij}$ as follows:

$$\bar{A} = \begin{bmatrix} \bar{A}_{q_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \bar{A}_{q_r} \end{bmatrix} + \begin{bmatrix} \bar{L}_1 \\ \vdots \\ \bar{L}_r \end{bmatrix}\bar{C},$$

$$\bar{W}_k = \begin{bmatrix} \bar{W}_{k11} & \cdots & \bar{W}_{k1r} \\ \vdots & \ddots & \vdots \\ \bar{W}_{kr1} & \cdots & \bar{W}_{krr} \end{bmatrix}, \; \bar{C} = \begin{bmatrix} \bar{C}_{q_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \bar{C}_{q_r} \end{bmatrix}.$$

A. Transformation to Canonical Form

It is in general not reasonable to expect the system (6) to match the canonical form (7). We would therefore like to transform (6) to the canonical form by applying nonsingular transformations to the state and output spaces. That is, we would like to find nonsingular matrices Γ_x and Γ_y, so that, by defining $x = \Gamma_x\chi$ and $y = \Gamma_y\gamma$, we obtain the canonical form (7), with $\bar{A} = \Gamma_x^{-1}A\Gamma_x$, $\bar{W}_k = \Gamma_x^{-1}W_k\Gamma_x$, $k = 1, \ldots, v$, and $\bar{C} = \Gamma_y^{-1}C\Gamma_x$. We now investigate when such transformations exist, and how they may be constructed.

The existing literature provides methods for finding transformations Γ_x and Γ_y that ensure the proper structure for $\bar{A}$ and $\bar{C}$. In particular, if we construct Γ_x and Γ_y to transform the observable triple $(C, A, 0)$ into the *special coordinate basis* (SCB) [17], then the matrices $\Gamma_x^{-1}A\Gamma_x$ and $\Gamma_y^{-1}C\Gamma_x$ will have the required structure (in general, a system in the SCB consists of four subsystems, but when applied to a triple an observable triple $(C, A, 0)$, only one of these subsystems have a nonzero dimension). Such transformations may therefore be found using available software for conversion to the SCB, written for *Matlab* [18] or *Maple* [19]. The question is now whether the proper structure for $\bar{W}_k$ can also be attained. It turns out that, if there exists any pair of transformations Γ_x and Γ_y that simultaneously gives $\bar{A}$, $\bar{C}$, and $\bar{W}_k$, $k = 1, \ldots, v$, the proper structure, then *every* pair of transformation that gives $\bar{A}$ and $\bar{C}$ the proper structure automatically gives $\bar{W}_k$, $k = 1, \ldots, v$, the proper structure. Thus we can construct Γ_x and Γ_y based only on A and C—if the resulting transformation does not satisfy the canonical form, then no nonsingular transformations will. This result is formalized in the next lemma, which also specifies precise conditions for when transformations exist, in terms of the interaction between the matrices A, C, and W_k, $k = 1, \ldots, v$.

Lemma 1: For each $i \in 1, \ldots, n$, let $\mathcal{S}_i = \ker[C^\top, (CA)^\top \; \cdots \; (CA^{i-1})^\top]^\top$. There exist nonsingular state and output transformations Γ_x and Γ_y that transform (6) into the canonical form (7) if, and only if, for each $i \in 1, \ldots, n$ and for each $k \in 1, \ldots, v$, the subspace $\mathcal{S}_i$ is W_k-invariant (i.e., $W_k\mathcal{S}_i \subset \mathcal{S}_i$). Furthermore, if this condition holds, then $\bar{W}_k = \Gamma_x^{-1}W_k\Gamma_x$ has the required form for all Γ_x and Γ_y that give $\bar{A} = \Gamma_x^{-1}A\Gamma_x$ and $\bar{C} = \Gamma_y^{-1}C\Gamma_x$ the required form.

Proof: See Appendix. ■

Example 1: Consider the system with $v = 1$ described by

$$A = \left[\begin{array}{cc|ccc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right], \; W_1 = \left[\begin{array}{cc|ccc} 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hline 0 & -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right], \; C = \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \end{array}\right].$$

Clearly, A and C already satisfy the required structure of the canonical form, but W_1 does not. Lemma 1 therefore implies that no nonsingular transformations can put the system in the canonical form. Indeed, it is easily verified that the condition $W_1\mathcal{S}_i \subset \mathcal{S}_i$ fails to hold for $i = 1$.

B. Observer Design

An observer design procedure for systems in the canonical form can be inferred from previous results that use similar canonical forms. For the sake of clarity and completeness, we nevertheless present a design procedure and a stability proof in this section. An observer for (6) is given by

$$\dot{\hat{\chi}} = \bar{A}\hat{\chi} + \sum_{k=1}^{v}\omega_k(t)\bar{W}_k\hat{\chi} + \bar{K}(\gamma - \bar{C}\hat{\chi}),$$

where $\bar{K}$ is the observer gain. By defining $\tilde{\chi} = \chi - \hat{\chi}$, we obtain the error dynamics

$$\dot{\tilde{\chi}} = \left(\bar{A} + \sum_{k=1}^{v}\omega_k(t)\bar{W}_k - \bar{K}\bar{C}\right)\tilde{\chi}. \tag{8}$$

The gain $\bar{K}$ is now selected to ensure global exponential stability of the origin of (8), as follows: For each $i = 1, \ldots, r$, let $K_i^\star = [K_{i1}^\star \; \cdots \; K_{iq_i}^\star]^\top$ be chosen such that the matrix $H_i := \bar{A}_{q_i} - K_i^\star\bar{C}_i$ is Hurwitz. Next, define $\bar{K}_i = [K_{i1}^\star/\varepsilon \; \cdots \; K_{iq_i}^\star/\varepsilon^{q_i}]^\top$, where $0 < \varepsilon \leq 1$ is a tuning parameter. Finally, define $\bar{K} = \mathrm{diag}(K_1, \ldots, K_r)$.

Lemma 2: If for each $k \in 1, \ldots, v$ the signals $\omega_k(t)$ are uniformly bounded by $|\omega_k(t)| \leq \omega_{k\max}$, then there exists an $0 < \varepsilon^\star \leq 1$ such that, for all $0 < \varepsilon \leq \varepsilon^\star$, the error dynamics (8) is globally exponentially stable.

Proof: We partition the error vector $\tilde{\chi}$ in the same way as χ, by writing $\tilde{\chi} = [\tilde{\chi}_1^\top \; \cdots \; \tilde{\chi}_r^\top]^\top$. For each $i = 1, \ldots, r$, define $\xi_i = \Theta_i\tilde{\chi}_i$, where $\Theta_i = \mathrm{diag}(1, \varepsilon, \ldots, \varepsilon^{q_i})$.

It is easily verified that the error dynamics in the new coordinates is given by

$$\varepsilon\dot{\xi}_i = H_i\xi_i + \varepsilon\sum_{k=1}^{v}\sum_{j=1}^{r}\omega_k(t)\Theta_i\bar{W}_{kij}\Theta_j^{-1}\xi_j, \quad i = 1,\ldots,r.$$

The lower-triangular structure of $\bar{W}_{kij}$ implies that there exists a bound M_{kij} such that, for all $0 < \varepsilon \le 1$, $\|\Theta_i\bar{W}_{kij}\Theta_j^{-1}\| \le M_{kij}$. For each $i = 1,\ldots,r$, let P_i be the unique symmetric positive-definite solution of the Lyapunov equation $P_iH_i + H_i^\top P_i = -I$, and consider the Lyapunov function candidate $V = \sum_{i=1}^{r}\xi_i^\top P_i\xi_i$. The time derivative of V is

$$\begin{aligned}\dot{V} &= -\sum_{i=1}^{r}\left(\|\xi_i\|^2 - 2\varepsilon\xi_i^\top P_i\sum_{k=1}^{v}\sum_{j=1}^{r}\omega_k(t)\Theta_i\bar{W}_{kij}\Theta_j^{-1}\xi_j\right)\\ &\le -\sum_{i=1}^{r}\left(\|\xi_i\|^2 - 2\varepsilon\|P_i\|\sum_{k=1}^{v}\sum_{j=1}^{r}\omega_{k\max}M_{kij}\|\xi_i\|\|\xi_j\|\right).\end{aligned}$$

This is a quadratic expression, where the indefinite terms vanish as $\varepsilon \to 0$. It is therefore straightforward to show that $\dot{V}$ is negative definite for all sufficiently small ε. ∎

C. Observer for the Original Nonlinear System

We have demonstrated how an observer may be designed for the linear time-varying system (7) in the canonical form. However, our goal is to design an observer of the form (3) for the nonlinear system (1), that guarantees stability of the error dynamics (4). We now show how to do this.

Suppose that we have found transformations Γ_x and Γ_y that take (6) to the canonical form (7), and that we have designed an observer gain matrix $\bar{K}$ according to the procedure described above. Now consider the observer (3) with gain $K = \Gamma_x\bar{K}\Gamma_y^{-1}$ and the resulting error dynamics (4). Using (5), the error dynamics (4) can be rewritten as

$$\dot{\tilde{x}} = \left(A + \sum_{k=1}^{v}\mu_k(t,x,\hat{x})W_k - KC\right)\tilde{x}.$$

Applying the transformation $\tilde{x} = \Gamma_x\tilde{\chi}$ and using the fact that $\Gamma_x^{-1}KC\Gamma_x = \Gamma_x^{-1}K\Gamma_y\bar{C} = \bar{K}\bar{C}$, we obtain the error dynamics (8) with $\omega_k(t) = \mu_k(t,x,\hat{x})$. Since the only assumption made about $\omega_k(t)$ is uniform boundedness, which is known to hold for $\mu_k(t,x,\hat{x})$ regardless of the trajectories of x and $\hat{x}$, we can conclude from Lemma 2 that the observer (3) with gain $K = \Gamma_x\bar{K}\Gamma_y^{-1}$ renders the origin of (4) globally exponentially stable, provided $\bar{K}$ is designed using a sufficiently low ε.

IV. Dynamic Output Shaping

Lemma 1 shows that whether the system (6) can be transformed to the canonical form by nonsingular transformations is a fundamental property of the matrices A, C, and W_k, $k = 1,\ldots,v$. One may easily conclude that, if the matrices do not satisfy this property, then nothing more can be done to achieve the canonical form. In many cases, however, it is possible to achieve the canonical form by increasing the number of integrations between the outputs and certain subspaces of the state space. To motivate this strategy, we consider another example.

Example 2: Consider the system with $v = 1$ described by

$$A = \left[\begin{array}{cc|ccc} 0&1&0&0&0\\ 0&0&0&0&0\\ \hline 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0\end{array}\right],\ W_1 = \left[\begin{array}{cc|ccc} 0&0&0&0&1\\ 0&0&0&0&0\\ \hline 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\end{array}\right],\ C = \left[\begin{array}{cc|ccc} 1&0&0&0&0\\ \hline 0&0&1&0&0\end{array}\right].$$

This system is used in [16, Ex. 3] as an example of a system that cannot be transformed to the canonical form by the method proposed therein. We see that the matrices A and C already satisfy the required structure of the canonical form, but W_1 does not. Suppose, however, that the output y_1 (i.e., the output from the second-order subsystem), is replaced by ξ_2, where $\dot{\xi}_2 = \xi_1$ and $\dot{\xi}_1 = y_1$ (i.e., a twice-integrated version of y_1). Then it is easy to confirm that the system does satisfy the canonical form.

Example 2 shows that introducing extra integrations may enable transformation to the canonical form, even when it is otherwise not possible. We shall now develop a systematic strategy for dynamically shaping the output y, with the goal of satisfying the invariance condition in Lemma 1.

Let $m \le n$, and suppose that for all $i \in 1,\ldots,m-1$, the condition $W_k\mathcal{S}_i \subset \mathcal{S}_i$ holds for all $k \in 1,\ldots,v$. The following algorithm checks whether the condition $W_k\mathcal{S}_m \subset \mathcal{S}_m$ also holds for all $k \in 1,\ldots,v$. If the condition does not hold, the algorithm extends the system by adding a stable filter to part of the output, thereby ensuring that the condition does hold for the extended system.

Algorithm 1: For $i = m-1$ and $i = m$, let $R_i = [C^\top \quad (CA)^\top \quad \cdots \quad (CA^{i-1})^\top]^\top$, and let $r_i = \operatorname{rank} R_i$ (if $m = 1$, then R_0 is an empty matrix and $r_0 = 0$). Clearly $R_m = [R_{m-1}^\top \quad (CA^{m-1})^\top]^\top$. Let therefore $S \in \mathbb{R}^{mp\times mp}$ be a nonsingular matrix such that

$$SR_m = \begin{bmatrix} R_{m-1}\\ R_m^\star\\ 0\end{bmatrix}, \quad S = \begin{bmatrix} I_{(m-1)p} & 0\\ 0 & S_{22}\\ S_{31} & S_{32}\end{bmatrix},$$

where $R_m^\star \in \mathbb{R}^{(r_m-r_{m-1})\times n}$ is of maximal rank $r_m - r_{m-1}$, $S_{22} \in \mathbb{R}^{(r_m-r_{m-1})\times p}$, $S_{31} \in \mathbb{R}^{(p+r_{m-1}-r_m)\times(m-1)p}$, and $S_{32} \in \mathbb{R}^{(p+r_{m-1}-r_m)\times p}$. Then $S_{22}CA^{m-1} = R_m^\star$ and $S_{32}CA^{m-1} = -S_{31}R_{m-1}$. Note that the choice of S is in general not unique.

Next, let the columns of E_0 be a linearly independent basis for the $(n-r_m)$-dimensional kernel of R_m. For $i = 1,\ldots,i'$, let the columns of E_i be a linearly independent basis for $\operatorname{im}[E_{i-1} \quad W_1E_{i-1} \quad \cdots \quad W_vE_{i-1}]$, where i' is the smallest integer such that $r_e := \operatorname{rank} E_{i'} = \operatorname{rank} E_{i'-1}$. Let $\ell = \operatorname{rank} R_m^\star E_{i'}$, and let $T \in \mathbb{R}^{(r_m-r_{m-1})\times(r_m-r_{m-1})}$ be a nonsingular matrix such that

$$TR_m^\star E_{i'} = \begin{bmatrix} 0\\ U\end{bmatrix}, \quad T = \begin{bmatrix} T_1\\ T_2\end{bmatrix}, \tag{9}$$

where $U = \mathbb{R}^{\ell\times r_e}$ is of maximal rank ℓ, $T_1 \in \mathbb{R}^{(r_m-r_{m-1}-\ell)\times(r_m-r_{m-1})}$, and $T_2 \in \mathbb{R}^{\ell\times(r_m-r_{m-1})}$. Note that the choice of T is in general not unique.

We now define a new, extended system by introducing the auxiliary state $\xi \in \mathbb{R}^{\ell}$, given by

$$\dot{\xi} = A_{\xi}\xi + T_2 S_{22} y,$$

where $A_{\xi} \in \mathbb{R}^{\ell \times \ell}$ is an arbitrarily chosen Hurwitz matrix. Thus ξ constitutes a filtered version of part of the output. We furthermore introduce a new output $y^{+} \in \mathbb{R}^{p}$, given by

$$y^{+} = \begin{bmatrix} \xi \\ S_{32} y \\ T_1 S_{22} y \end{bmatrix}.$$

Gathering the new states and the original states in a single vector $x^{+} = [\xi^{\top} \quad x^{\top}]^{\top}$, we obtain the extended system description

$$\dot{x}^{+} = A^{+}x^{+} + \sum_{k=1}^{v} \omega_k(t) W_k^{+} x^{+}, \quad y^{+} = C^{+}x^{+},$$

where

$$A^{+} = \begin{bmatrix} A_{\xi} & T_2 S_{22} C \\ 0 & A \end{bmatrix}, \; W_k^{+} = \begin{bmatrix} 0 & 0 \\ 0 & W_k \end{bmatrix},$$
$$C^{+} = \begin{bmatrix} I & 0 \\ 0 & S_{32} C \\ 0 & T_1 S_{22} C \end{bmatrix}.$$

Lemma 3: Suppose that for all $i \in 1, \ldots, m-1$, the invariance condition $W_k \mathcal{S}_i \subset \mathcal{S}_i$ holds for all $k \in 1, \ldots, v$, and that Algorithm 1 is executed. Then the invariance condition $W_k^{+} \mathcal{S}_m^{+} \subset \mathcal{S}_m^{+}$ holds for all $k \in 1, \ldots, v$, where $\mathcal{S}_m^{+}$ is defined in the same way as $\mathcal{S}_m$, but with respect to the extended pair (C^{+}, A^{+}).

Proof: See Appendix. ■

Algorithm 1 ensures that the invariance condition of Lemma 1 holds for $i = m$ for the extended system, provided the invariance condition holds for $i = 1, \ldots, m-1$ for the original system. Crucially, it can be shown that the extension of the state space that results from Algorithm 1 is minimal, in the sense that no lower-order extension could make the extended system satisfy the invariance condition for $i = m$ without discarding part of the output. Consequently, no extension takes place if the original system already satisfies the invariance requirement for $i = m$. One might hope that, by simply executing Algorithm 1 for $m = 1, 2, \ldots$, one would eventually end up with a system for which the full invariance requirement in Lemma 1 holds. Unfortunately, this is not the case in general. Even though Algorithm 1 ensures that the extended system satisfies the invariance condition for $i = m$, it may happen that it does not satisfy the condition for some $i < m$, even though this condition was satisfied by the original system. Thus, if executing Algorithm 1 results in an extension of the state space for a given m, we need to go back to $m = 1$ and check every subspace again. This leads to the following algorithm.

Algorithm 2: Execute Algorithm 1 for $m = 1, 2, \ldots$ If for any m, Algorithm 1 results in an extension of the state space, start over again by running Algorithm 1 on the extended system for $m = 1, 2, \ldots$. Continue in this fashion until either (i) Algorithm 1 has been executed for each $m = 1, 2, \ldots$ up to the order of the current system without a resulting extension of the state space; or (ii) the state space has been extended more than n^2 times. In the first case, Algorithm 2 terminates successfully; in the second case, it terminates unsuccessfully.

It is clear that, if Algorithm 2 terminates successfully, then the resulting system (which may have been extended multiple times) satisfies the full invariance requirement of Lemma 1. Thus it can be transformed to the canonical form (7) by using already available methods. Although Algorithms 1 and 2 are too complicated to be carried out by hand in most cases, they can be implemented in software. Indeed, we have implemented a preliminary version of Algorithm 2 in Maple, by using the same techniques as those used in [19]. We shall now illustrate the effectiveness of this implementation on two simple examples.

Example 3: Consider again the system in Example 1. As already discussed, this system cannot be transformed to the canonical form by introducing nonsingular transformations of the state and output spaces. Moreover, the system does not satisfy any of the canonical forms from the literature referenced in the introduction. When we execute our implementation of Algorithm 2, Algorithm 1 is first executed for $m = 1$, which results in the following extension:

$$\dot{\xi} = -\xi + \begin{bmatrix} 0 & 1 \end{bmatrix} y, \quad y^{+} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \xi \\ y \end{bmatrix}.$$

After this, Algorithm 1 is executed on the extended system for $m = 1, \ldots, 6$ without any resulting extensions. Thus, the invariance condition of Lemma 1 is satisfied, and the system is transformed into the canonical form (7), with

$$\bar{A} = \left[\begin{array}{ccc|ccc} -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \bar{W}_1 = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right],$$

and $\bar{C}$ given in the obvious way according to the canonical form. An observer can now be designed by the method described in Section III-B. Due to space constraints, we have not included the transformations Γ_x and Γ_y here.

Example 4: Consider again the system in Example 2. As already discussed, this system is used as an example in [16] of a system that cannot be handled by the method proposed therein. When we execute our implementation of Algorithm 2, Algorithm 1 is first executed for $m = 1$, which results in the following extension:

$$\dot{\xi}_1 = -\xi_1 + \begin{bmatrix} 1 & 0 \end{bmatrix} y, \quad y^{+} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ y \end{bmatrix}.$$

Next, Algorithm 1 is executed on the extended system for $m = 1$, without a resulting extension, and for $m = 2$, which results in the following extension:

$$\dot{\xi}_2 = -\xi_2 + \begin{bmatrix} 1 & 0 \end{bmatrix} y^{+}, \quad y^{++} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} y^{+},$$

(We have used a double superscript to indicate that y^{++} is the output after two extensions have taken place.) Next, Algorithm 1 is executed on the twice-extended system for $m = 1, \ldots, 7$, without any resulting extensions. Thus, the invariance condition of Lemma 1 is satisfied, and the system is transformed into the canonical form (7), with

$$\bar{A} = \left[\begin{array}{ccc|ccc} 0\,1\,0 & 0 & 0\,0\,0 \\ 0\,0\,1 & 0 & 0\,0\,0 \\ 0\,0\,0 & 0 & 0\,0\,0 \\ \hline 0\,0\,0 & -2 & 1\,0\,0 \\ 0\,0\,0 & -1 & 0\,1\,0 \\ 0\,0\,0 & 0 & 0\,0\,1 \\ 0\,0\,0 & 0 & 0\,0\,0 \end{array}\right], \; \bar{W}_1 = \left[\begin{array}{c|c} 0\,0\,0 & 0\,0\,0\,0 \\ 0\,0\,0 & 0\,0\,0\,0 \\ 0\,0\,0 & 0\,0\,0\,0 \\ \hline 0\,0\,0 & 0\,0\,0\,0 \\ 0\,0\,0 & 0\,0\,0\,0 \\ 0\,0\,1 & 0\,0\,0\,0 \\ 0\,0\,0 & 0\,0\,0\,0 \end{array}\right],$$

and $\bar{C}$ given in the obvious way according to the canonical form. An observer can now be designed by the method described in Section III-B. Again, we have not included the matrices Γ_x and Γ_y, due to space constraints. We remark that, although we have chosen examples for which a clear internal structure is visible to the naked eye, our implementation of Algorithm 2 executes with the same success when these and other systems are transformed to alternative, random coordinate bases, where no such structure is visible.

Remark 1: Even though we have focused on extending the system (6), the ultimate goal is to design observers for the nonlinear system (1) from which (6) is derived. Indeed, if the proper output shaping can be constructed to transform (6) to the canonical form and an observer gain $\bar{K}$ is found by the procedure in Section III-B, then the same shaping can be applied to the original system and an observer implemented as described in Section III-C.

A. Unsuccessful Termination

We have proven that if Algorithm 2 terminates successfully, then the resulting system can be transformed to the canonical form (7), and thus an observer can be designed by the method described in Section III-B. We have furthermore demonstrated the effectiveness of the algorithm on two examples. What we have not yet addressed is the prospect of unsuccessful termination. It is our conjecture that the algorithm terminates unsuccessfully only if the conditions of Lemma 1 can never be satisfied by extensions of the type proposed in this paper. This conjecture is supported by tests conducted using the Maple implementation of the algorithm. Nevertheless, we are not yet able to provide a complete proof of this conjecture.

V. Concluding Remarks

We have proposed a strategy of dynamic output shaping as a method for enabling transformation to a particular canonical form, and we have presented a constructive algorithm for designing the required filters. Nevertheless, much remains to be done. In particular, we are working on a proof of our conjecture, which essentially says that the algorithm finds a solution whenever a solution exists.

It can be shown that a slightly larger class of systems can be handled if, instead of replacing parts of the output with filtered outputs, the output vector is augmented with filtered outputs while retaining the original outputs. This extension will be presented in an expanded version of this paper.

As we have have already mentioned, the canonical form used in this paper is similar or equivalent to canonical forms used in several other places in the literature. Nevertheless, several different variations of this form are found throughout the literature, and it is of interest to investigate whether these can in fact be converted to the canonical form (7) by dynamically shaping the outputs.

Ultimately, our goal is to produce a computer-implementable algorithm that, for a large class of systems of the form (1), can answer the question of whether a nonlinearity can be dominated in an observer of the form (3), and that can automatically synthesize gains to achieve such domination.

Appendix

Proof of Lemma 1: As explained above, there always exists a pair of transformations that give $\bar{A}$ and $\bar{C}$ the required form. Thus, we must show that (i) if for such a pair of transformations, $\bar{W}_k$ also has the required form for all $k \in 1, \ldots, v$, then for all $i \in 1, \ldots, n$ and $k \in 1, \ldots, v$, $\mathcal{S}_i$ is W_k-invariant; and (ii) if for all $i \in 1, \ldots, n$ and $k \in 1, \ldots, v$, $\mathcal{S}_i$ is W_k-invariant, then $\bar{W}_k = \Gamma_x^{-1} W_k \Gamma_x$ always has the required form.

Suppose first that $\bar{W}_k$ has the required form, so that the transformed system is described by the canonical form (7). For each $i \in 1, \ldots, n$ and $j \in 1, \ldots, r$, let $\chi_{ji}^\star$ consist of the first i elements of χ_j, or all the elements of χ_j if $q_j \leq i$. Furthermore, define $\chi_i^\star$ by stacking $\chi_{1i}^\star, \ldots, \chi_{ri}^\star$. Let $\bar{R}_i = [\bar{C}^\top \;\; (\bar{C}\bar{A})^\top \;\; \cdots \;\; (\bar{C}\bar{A}^{i-1})^\top]^\top$, and note that $\ker \bar{R}_i = \Gamma_x^{-1}\mathcal{S}_i$. From the canonical form (7), it is straightforward to see that $\chi_{ji}^\star$ consists of $\bar{C}_j\chi_j, \bar{C}_j\bar{A}_j\chi_j, \ldots, \bar{C}_j\bar{A}_j^{i-1}\chi_j$ and, hence, that $\chi_i^\star$ is a combination of $\bar{C}\chi, \bar{C}\bar{A}\chi, \ldots, \bar{C}\bar{A}^{i-1}\chi$. That is, $\chi_i^\star = \Lambda_i\chi$, where Λ_i has the same rank as $\bar{R}_i$ and the rows of Λ_i are combinations of the rows of $\bar{R}_i$ (thus, $\ker \Lambda_i = \ker \bar{R}_i = \Gamma_x^{-1}\mathcal{S}_i$). The lower-triangular structure of each matrix $\bar{W}_{kij}$ specifies that the time-varying influence in the derivative of each $\chi_{ji}^\star$ must only depend on states higher up in each integrator chain. Put differently, the time-varying influence in the derivative of $\chi_i^\star$ must only depend on $\chi_i^\star$. Since $\chi_i^\star = \Lambda_i\chi$, this requirement can only be satisfied if there exists a matrix L_{ik} such that $\Lambda_i\bar{W}_k\chi = L_{ik}\Lambda_i\chi$, which can also be written as $\bar{W}_k^\top\Lambda_i^\top = \Lambda_i^\top L_{ik}^\top$. This implies that $\operatorname{im}\Lambda_i^\top$ must be $\bar{W}_k^\top$-invariant, which in turn implies that $\ker\Lambda_i = \Gamma_x^{-1}\mathcal{S}_i$ must be $\bar{W}_k$-invariant (see, e.g., [20]). Thus, $\mathcal{S}_i$ must be W_k-invariant. This argument holds for all $i \in 1, \ldots, n$ and $k \in 1, \ldots, v$.

Conversely, suppose that $\mathcal{S}_i$ is W_k-invariant. We need to show that $\bar{W}_k$ has the required form, which specifies that for all $i \in 1, \ldots, n$, the time-varying term $\omega_k(t)\Lambda_i\bar{W}_k\chi$, which occurs in the derivative of $\chi_i^\star$, can only depend on $\chi_i^\star$. Since $\mathcal{S}_i$ is W_k-invariant, we know that $\Gamma_x^{-1}\mathcal{S}_i = \ker\Lambda_i$ is $\bar{W}_k$-invariant. This furthermore implies that $\operatorname{im}\Lambda_i^\top$ is $\bar{W}_k^\top$-invariant; hence, there exists an L_{ik} such that $\bar{W}_k^\top\Lambda_i^\top = \Lambda_i^\top L_{ik}^\top$, which can also be written as $\Lambda_i\bar{W}_k = L_{ik}\Lambda_i$. Thus, we have $\omega_k(t)\Lambda_i\bar{W}_k\chi = \omega_k(t)L_{ik}\Lambda_i\chi = \omega_k(t)L_{ik}\chi_i^\star$, which means that $\bar{W}_k$ has the required form. □

Proof of Lemma 3: Let R_m^+ be defined in the same way as R_m, but with respect to the pair (C^+, A^+), such that

$\mathcal{S}_m^+ = \ker R_m^+$. We must show that $W_k^+ \mathcal{S}_m^+ \subset \mathcal{S}_m^+$ for all $k \in 1, \ldots, v$. It is easily verified that for $A_\xi = 0$, we have

$$R_m^+ = \begin{bmatrix} I & 0 \\ 0 & MR_{m-1} \\ 0 & S_{32}CA^{m-1} \\ 0 & T_1 S_{22} CA^{m-1} \end{bmatrix}, \tag{10}$$

where M is a nonsingular block-diagonal matrix with the blocks $[S_{32}^\top \quad (TS_{22})^\top]^\top$ along the diagonal. It also easy (although slightly tedious) to show that $\ker R_m^+$ remains the same regardless of A_ξ. From (10) we therefore see that $x^+ = [\xi^\top \quad x^\top]^\top \in \mathcal{S}_m^+$ is equivalent to $\xi = 0$ and

$$x \in \ker \begin{bmatrix} MR_{m-1} \\ S_{32}CA^{m-1} \\ T_1 S_{22} CA^{m-1} \end{bmatrix} = \ker \begin{bmatrix} MR_{m-1} \\ -S_{31} R_{m-1} \\ T_1 R_m^\star \end{bmatrix} = \ker \begin{bmatrix} R_{m-1} \\ T_1 R_m^\star \end{bmatrix}.$$

It follows that the invariance condition $W_k^+ \mathcal{S}_m^+ \subset \mathcal{S}_m^+$ is equivalent to the condition $x \in \ker [R_{m-1}^\top \quad (T_1 R_m^\star)^\top]^\top \implies W_k x \in \ker [R_{m-1}^\top \quad (T_1 R_m^\star)^\top]^\top$. Thus, we must show that $\ker [R_{i-1}^\top \quad (T_1 R_i^\star)^\top]^\top$ is W_k-invariant for all $k \in 1, \ldots, v$. We can show that $\operatorname{im} E_{i'}$ is the smallest subspace containing $\ker R_m$ that is W_k-invariant for all $k \in 1, \ldots, v$. [2] We denote this subspace by $\mathcal{V}_m$ and define $\mathcal{V}_{m-1}$ in the same way with respect to R_{m-1}. We shall show that $\ker [R_{m-1}^\top \quad (T_1 R_m^\star)^\top]^\top = \mathcal{V}_m$, which will complete the proof.

Since $\mathcal{S}_{m-1} = \ker R_{m-1}$ is W_k-invariant for all $k \in 1, \ldots, v$, we know that $\ker R_{m-1} = \mathcal{V}_{m-1} \supset \mathcal{V}_m$. It follows that $\operatorname{im} R_{m-1}^\top \perp \mathcal{V}_m$, and hence $R_{m-1} E_{i'} = 0$. From (9), we also see that $T_1 R_m^\star E_{i'} = 0$. Hence, $[R_{m-1}^\top \quad (T_1 R_m^\star)^\top]^\top E_{i'} = 0$, which implies that $\mathcal{V}_m \subset \ker [R_{m-1}^\top \quad (T_1 R_m^\star)^\top]^\top$. If we can also show that $\dim \ker [R_{m-1}^\top \quad (T_1 R_m^\star)^\top]^\top \leq \dim \mathcal{V}_m$, then clearly $\ker [R_{m-1}^\top \quad (T_1 R_m^\star)^\top]^\top = \mathcal{V}_m$, as desired.

We have

$$\begin{aligned} \dim \mathcal{V}_m &= \dim \mathcal{V}_m \cap \ker R_m + \dim \mathcal{V}_m \cap \operatorname{im} R_m^\top \\ &= \dim \ker R_m + \dim \operatorname{im} E_{i'} \cap \operatorname{im} R_m^\top. \end{aligned}$$

Since $\ker R_m \subset \operatorname{im} E_{i'}$, we have $\dim \operatorname{im} E_{i'} \cap \operatorname{im} R_m^\top = \operatorname{rank} R_m E_{i'} = \ell$. Hence, $\dim \mathcal{V}_m = \dim \ker R_m + \ell$. On the other hand, we have

$$\begin{aligned} \dim \ker \begin{bmatrix} R_{m-1} \\ T_1 R_m^\star \end{bmatrix} &\leq \dim \ker \begin{bmatrix} R_{i-1} \\ T_1 R_m^\star \\ T_2 R_m^\star \end{bmatrix} + \operatorname{rank} T_2 R_m^\star \\ &= \dim \ker R_m + \ell. \end{aligned}$$

Thus, we see that $\dim \ker [R_{m-1}^\top \quad (T_1 R_m^\star)^\top]^\top \leq \dim \mathcal{V}_m$, as desired.

That the pair (C^+, A^+) is observable follows from observing that

$$\ker R_{n+1}^+ = \ker \begin{bmatrix} I & 0 \\ 0 & MR_n \\ 0 & S_{32}CA^n \\ 0 & T_1 S_{22} CA^n \end{bmatrix} = \ker \begin{bmatrix} I & 0 \\ 0 & R_n \end{bmatrix} = 0,$$

where we have used the fact that (C, A) is observable to conclude that $\ker R_n = 0$. □

[2] From the definition of $E_{i'}$ note that $W_k \operatorname{im} E_{i'-1} \subset \operatorname{im} E_{i'}$, and that $\operatorname{im} E_{i'} = \operatorname{im} E_{i'-1}$. Thus $W_k \operatorname{im} E_{i'} \subset \operatorname{im} E_{i'}$. Since $\operatorname{im} E_0 = \ker R_m$, it is clear that $\ker R_m \subset \operatorname{im} E_{i'}$. Finally, if $\mathcal{N}$ is a W_k-invariant subspace for all $k \in 1, \ldots, v$, which contains $\ker R_m$, then we must have $\sum_{k=1}^{v} (W_k \operatorname{im} E_0) = \operatorname{im} E_1 \subset \mathcal{N}$, $\sum_{k=1}^{v} (W_k \operatorname{im} E_1) = \operatorname{im} E_2 \subset \mathcal{N}$, and so on. Hence $\operatorname{im} E_{i'} \subset \mathcal{N}$.

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