Homogeneous Networks of Non-Introspective Agents Under External Disturbances - H_∞ Almost Synchronization

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Abstract

This paper addresses the problem of "H_∞ almost synchronization" for homogeneous networks of general linear agents subject to external disturbances and under directional communication links. Agents are presumed to be *non-introspective*; i.e. agents are not aware of their own states or outputs, and the only available information for each agent is a network measurement that is a linear combination of relative outputs. Under a certain set of conditions, a family of dynamic protocols is developed such that the impact of disturbances on the synchronization error dynamics, expressed in terms of the H_∞ norm of the corresponding closed-loop transfer function, is reduced to any arbitrarily small value.

Key words: Synchronization, Disturbance rejection, H_∞ robust control, Homogeneous Multi-agent systems.

1 Introduction

The problem of synchronization in networks of dynamic systems has attracted enourmous attention in recent years and possesses numerous applications in areas such as formation flying, cooperative control, and distributed sensor fusion; see e.g. (Olfati-Saber et al., 2007) and references therein. The objective of synchronization in multi-agent systems is to find a distributed algorithm to reach an agreement on a certain quantity of interest which depends on the state of agents.

The seminal works of (Wu & Chua, 1995a) and (Wu & Chua, 1995b) have substantially contributed in analysis and design of multi-agent systems by introducing the application of the graph theory and the Kronecker product. Moreover, the works of Jadbabaie et al. (2003); Olfati-Saber & Murray (2004); Fax & Murray (2004); Moreau (2005), and Ren & Beard (2005) have been instrumental in paving the way that synchronization protocols have been developed. Also, Li et al. (2010) extended conventional observers to distributed observers with the aid of allowing agents to exchange their protocol's states. A thorough coverage of earlier work, including static and dynamic protocols, the effect of communication delay, and dynamic interaction topologies may be found in (Wu, 2007), (Ren & Cao, 2011), and (Cao et al., 2013) and references therein. A recent research on output synchronization of heterogeneous networks was presented by Grip et al. (2012).

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1.1 *The Topic of This Paper*

In (Peymani et al., 2014), we introduced the notion of H_∞ almost synchronization [2], in which the impact of external disturbances on the disagreement dynamics is attenuated to any arbitrarily small value in the sense of the H_∞ norm of the closed-loop transfer function. Hence, synchronization can be achieved with any degree of accuracy. The work of Peymani et al. (2014) was concerned with *heterogeneous* networks of *introspective* agents; i.e. the dynamics of agents are non-identical, and agents have partial knowledge about their own states in addition to network measurements, which are based on relative information.

In this article, the problem of H_∞ almost synchronization is solved for *homogeneous* networks of *non-introspective* agents; i.e. the network consists of identical agents which are *not* allowed to access their own states or outputs, and the only available measurement given to each agent is a linear combination of the output of the agent relative to that of its own neighbors. A practical motivation is a swarm of autonomous underwater vehicles which do not measure their absolute positions, but can exchange their relative distances.

We stress the fact that lack of self-measurements does not allow us to shape the agents into the desired dynamics, as proposed by Peymani et al. (2014). Hence, we are confronted with general linear systems, where the finite and infinite zero structures as well as invertibility properties are explicitly exploited in order to achieve H_∞ almost synchronization.

[2] Historically, the term "almost" has been applied to the problem of finding families of controllers that can reduce noise sensitivity to any arbitrary level; see e.g. (Ozcetin et al., 1992) where the problem of almost disturbance decoupling was addressed.

Our design method is based on the time-scale structure assignment technique (Ozcetin et al., 1992) rooted in the methodology of singular perturbation (Kokotovic et al., 1986). We propose a novel time-scale structure assignment technique where infinite zero dynamics of different orders are scaled with different time-scales. The motivation for such a time-scale assignment is to make the design robust with respect to the multiplicative uncertainty arising from the interaction topology.

1.2 Other Related Work

Although synchronization of multi-agent systems has been vastly studied in ideal conditions, relatively few references concern synchronization in the presence of disturbances. They mainly focus on homogeneous multi-agent systems and utilize linear matrix inequalities to solve the H_∞ optimization problem over a network. Some only analyze the disturbance attenuation properties of consensus protocols and do not present a specific design procedure to solve the problem of disturbance rejection in multi-agent systems.

Under bidirectional communication links, according to (Li et al., 2010), the distributed H_∞ control problem to achieve a desired H_∞ gain for the transfer function from disturbance to each agent's output is converted to the H_∞ control problem of a set of independent systems. Using the pinning idea, which implies a subset of agents have access to its own states, Li et al. (2011b) introduced the notions of H_2/H_∞ performance regions, and proposed a static protocol that assured an unbounded H_∞ performance region. The works of Li et al. (2010, 2011b) did not explore disturbance attenuation from synchronization error dynamics. Shen et al. (2011) introduced the notion of the bounded H_∞ synchronization, and Wang et al. (2013) defined the H_∞ performance over a finite horizon for a class of discrete time-varying multi-agent systems subject to missing measurements and parameter uncertainties. For complete and circulant networks, Massioni & Verhaegen (2009) proposed a decomposition approach to solve the distributed H_∞ control problem for homogenous multi-agent systems. For networks of non-introspective integrators and introspective linear systems with full-state coupling, Mo & Jia (2011) and Liu & Jia (2011) presented static controllers to solve the H_∞ control problem over a network.

For directed networks of first-order integrator systems, Lin et al. (2008) proposed static H_∞ controllers; extensions to networks of introspective high-order multi-agent systems were given in (Lin & Jia, 2010; Liu & Jia, 2010). For networks of introspective single-integrators, Hong-Yong et al. (2011) proposed a consensus protocol by designing an observer for disturbances generated by an exosystem. Considering networks of non-introspective agents possessing Lipschitz nonlinear dynamics and exchanging full-state information according to communication topologies described by strongly connected and balanced directed graphs, Li et al. (2012) proposed a static protocol, which guaranteed a desired H_∞ performance. Moreover, Shen et al. (2010) and Ugrinovskii (2011) studied the topic of H_∞ distributed consensus filtering.

Besides LMI-based H_∞ consensus protocols, the following articles investigate disturbance attenuation properties of consensus protocols. The notion of leader-to-formation stability (Tanner et al., 2004) was proposed to assess robustness of followers with respect to the leader's input for nonlinear agents. Li & Zhang (2009) solved unbiased mean-square average-consensus by introducing time-varying consensus gains. For first-order dynamical systems under bidirectional links where measurements are corrupted by bounded noise, Bauso et al. (2009) proposed a static controller which guaranteed convergence of all states to a cylinder. Using non-smooth finite-time consensus algorithms for networks of double-integrators, Li et al. (2011a) and Du et al. (2012) provided an analysis for disturbance attenuation property of the closed-loop system in the presence of external disturbances.

Hence, one observes a distinct lack of systematic approaches to design synchronization protocols for networks of non-introspective, general linear systems, coupled through partial state information and under directional communication links, which are capable of synchronizing the agents with arbitrary accuracy in the presence of external disturbances; this article aims to fill this gap.

1.3 Notations

Throughout the paper, matrix A is represented by $A=[a_{ij}]$ where the element (i,j) of A is shown by a_{ij}. Ker$A$ and ImA denote respectively the kernel and the image of A. The Kronecker product of matrices $A=[a_{ij}]$ and $B=[b_{ij}]$ is defined as $A\otimes B=[a_{ij}B]$. Let $\|A\|$ denote the induced 2-norm. A block diagonal matrix constructed by A_i's is shown by diag$\{A_i\}$ for $i=1,\cdots,n$. Also, stack$\{A_i\}$ for $i=1,\cdots,n$ indicates $[A_1^{\mathrm{T}},A_2^{\mathrm{T}},\cdots,A_n^{\mathrm{T}}]^{\mathrm{T}}$, and $x=\mathrm{col}\{x_i\}$ for $i=1,\cdots,n$ is adopted to denote $x=[x_1^{\mathrm{T}},\cdots,x_n^{\mathrm{T}}]^{\mathrm{T}}$ where x_i's are vectors.

The identity matrix of order n is symbolized by I_n. Let $\mathbf{1}_n\in\mathbb{R}^n$ be the vector with all entries equal to one. The real part of a complex number λ is represented by Re$\{\lambda\}$. The open left-half and the open right-half complex planes are represented by $\mathbb{C}^-$ and $\mathbb{C}^+$, respectively. The H_∞ norm of a transfer function $T(s)$ is denoted $\|T(s)\|_\infty$. For a space $\mathscr{V}$, the orthogonal complement is shown by $\mathscr{V}^\perp$.

2 Homogeneous Multi-Agent Systems

A homogeneous multi-agent system is referred to a network of identical multi-input multi-output agents described by

$$\text{Agent } i: \dot{\mathbf{x}}_i=\mathbf{A}\mathbf{x}_i+\mathbf{B}\mathbf{u}_i+\mathbf{G}\mathbf{w}_i, \qquad \mathbf{y}_i=\mathbf{C}\mathbf{x}_i \tag{1a}$$

in which $i\in\mathfrak{G}\triangleq\{1,\cdots,N\}$. Also, $\mathbf{x}_i\in\mathbb{R}^n$ is the state, $\mathbf{u}_i\in\mathbb{R}^m$ is the control, $\mathbf{y}_i\in\mathbb{R}^p$ is the output, $\mathbf{w}_i\in\mathbb{R}^\omega$: $\lim_{\tau\to\infty}\frac{1}{2\tau}\int_{-\tau}^{\tau}\mathbf{w}_i^{\mathrm{T}}\mathbf{w}_i\,\mathrm{d}t<\infty$ is the external disturbance.

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The network's communication topology, based on which agents exchange information, is described by a directed graph $\mathscr{G}$ whose nodes correspond with agents in the network. If an edge exists from node k to node i, a positive real weight a_{ik} is given to the edge. We assume that no self loops are allowed; i.e. $a_{ii} = 0$. The graph $\mathscr{G}$ is associated with the Laplacian matrix $\mathsf{G} = [g_{ik}]$ where $g_{ik} = -a_{ik}$ for $i,k \in \mathfrak{S}$, $i \neq k$ and $g_{ii} = \sum_{k=1,k\neq i}^{N} a_{ik}$. It follows that $\lambda = 0$ is an eigenvalue of G with a right eigenvector $\mathbf{1}_N$. Thus, the network measurement given to agent $i \in \mathfrak{S}$ is:

$$\zeta_i = \sum_{k=1}^{N} a_{ik}(\mathbf{y}_i - \mathbf{y}_k) = \sum_{k=1}^{N} g_{ik}\mathbf{C}\mathbf{x}_k \tag{1b}$$

In addition, it is assumed that agents are capable of exchanging additional information over the network. The transmission of this additional information conforms with the network's communication topology and facilitates the design of a distributed observer for the network. Thus, agent i has access to the following quantity:

$$\hat{\zeta}_i = \sum_{k=1}^{N} g_{ik}\eta_k \tag{1c}$$

where $\eta_k \in \mathbb{R}^p$ depends on the state of the protocol of agent $k \in \mathfrak{S}$; it will be clarified later when a dynamic protocol is introduced. Agents are non-introspective and no self measurements are available; in other words, agent i does not have access to its own states $\mathbf{x}_i$ or the output $\mathbf{y}_i$. Thus, the only information available to agent i is the network measurement ζ_i, which is based on relative information, and neither $\mathbf{x}_i$ nor $\mathbf{y}_i$ is measured.

3 H_∞ Almost Synchronization

We define the following vectors which are formed by stacking the corresponding vectors of each agent:

$$w \triangleq \text{col}\{\mathbf{w}_i\}, \qquad \zeta \triangleq \text{col}\{\zeta_i\} \tag{2}$$

for $i \in \mathfrak{S}$. Let the mutual disagreement be defined as:

$$\mathfrak{e}_{i,k} \triangleq \mathbf{y}_i - \mathbf{y}_k, \quad \text{for } i,k \in \mathfrak{S},\ i > k \tag{3}$$

The stacking column vector of all mutual disagreements is denoted $\mathfrak{e}$. Obviously, synchronization is achieved if $\mathfrak{e} = 0$. We define the following transfer function with the appropriate dimension: $\mathfrak{e} = T_{w\mathfrak{e}}(s)w$. The problem that we cope with is precisely stated in Problem 1.

Problem 1 *Consider a multi-agent system as described by (1) with a communication topology $\mathscr{G}$. Given a set of network graphs $\mathscr{G}^*$ and any $\gamma > 0$, the "H_∞ almost synchronization" problem is to find, if possible, a linear time-invariant dynamic protocol such that, for any $\mathscr{G} \in \mathscr{G}^*$, the closed-loop transfer function from w to $\mathfrak{e}$ satisfies $\|T_{w\mathfrak{e}}(s)\|_\infty < \gamma$.*

For agent $i \in \mathfrak{S}$, the protocol, whose internal state is denoted $\xi_i \in \mathbb{R}^{\mathfrak{q}}$ for some integer $\mathfrak{q} > 0$, maps ζ_i and $\hat{\zeta}_i$ to $\mathbf{u}_i$ and takes the following general form

$$\begin{cases} \dot{\xi}_i = \mathscr{A}_c(\varepsilon)\,\xi_i + \mathscr{B}_c(\varepsilon)\,\text{col}\{\zeta_i, \hat{\zeta}_i\} & \text{(4a)} \\ \mathbf{u}_i = \mathscr{C}_c(\varepsilon)\,\xi_i + \mathscr{D}_c(\varepsilon)\,\text{col}\{\zeta_i, \hat{\zeta}_i\} & \text{(4b)} \end{cases}$$

The matrices $\mathscr{A}_c(\varepsilon)$, $\mathscr{B}_c(\varepsilon)$, $\mathscr{C}_c(\varepsilon)$, and $\mathscr{D}_c(\varepsilon)$ are parameterized in terms of the tuning parameter $0 < \varepsilon \leq 1$. We will specify these matrices in the subsequent subsections. We will find an upper bound for ε, say ε^*, such that for any $\varepsilon \leq \varepsilon^*$, we obtain $\|T_{w\mathfrak{e}}(s)\|_\infty < \gamma$. The controller is continuous in ε, and adjustment of ε may be carried out online to obtain the required accuracy of synchronization. Thus, it turns out to a non-iterative (one-shot) design.

3.1 Preliminaries and Assumptions

The conditions under which the development of the desired protocols is viable are given using the concepts from the geometric control theory and in terms of an appropriate set of network graphs. The geometric control theory and its application to exact disturbance decoupling are discussed in the books of Wonham (1985) and Trentelman et al. (2001). The application to almost disturbance decoupling is presented by Weiland & Willems (1989).

Let $\mathscr{V}^*_{\text{Ker}\,\mathbf{C}}(\mathbf{A},\mathbf{B},\mathbf{C})$ be the maximal $(\mathbf{A}-\mathbf{BF})$-invariant subspace of $\mathbb{R}^n$ contained in $\text{Ker}\,\mathbf{C}$ such that the eigenvalues of $(\mathbf{A}-\mathbf{BF})$ belong to $\mathbb{C}^-$ for some $\mathbf{F}$. The supremal $\mathscr{L}_p$-almost controllability subspace 'contained' in $\text{Ker}\,\mathbf{C}$ is represented by $\mathscr{R}^*_{\text{Ker}\,\mathbf{C}}(\mathbf{A},\mathbf{B},\mathbf{C})$. Let $\mathscr{S}^*_{\text{Im}\,\mathbf{B}}(\mathbf{A},\mathbf{B},\mathbf{C})$ denote the minimal $(\mathbf{A}-\mathbf{KC})$-invariant subspace of $\mathbb{R}^n$ containing $\text{Im}\,\mathbf{B}$ such that the eigenvalues of $(\mathbf{A}-\mathbf{KC})$ belong to $\mathbb{C}^-$ for some $\mathbf{K}$. We define the following subspaces of the state space.

$$\mathscr{V}_{b,\text{Ker}\,\mathbf{C}}(\mathbf{A},\mathbf{B},\mathbf{C}) = \mathscr{V}^*_{\text{Ker}\,\mathbf{C}}(\mathbf{A},\mathbf{B},\mathbf{C}) \oplus \mathscr{R}^*_{\text{Ker}\,\mathbf{C}}(\mathbf{A},\mathbf{B},\mathbf{C})$$
$$\mathscr{S}_{b,\text{Im}\,\mathbf{B}}(\mathbf{A},\mathbf{B},\mathbf{C}) = (\mathscr{V}_{b,\text{Ker}\,\mathbf{B}^{\text{T}}}(\mathbf{A}^{\text{T}},\mathbf{C}^{\text{T}},\mathbf{B}^{\text{T}}))^{\perp}$$

Assumption 1 *We make the following assumptions.*

(1) $(\mathbf{A},\mathbf{B})$ *is stabilizable, and* $(\mathbf{C},\mathbf{A})$ *is detectable;*
(2) $\text{Im}\,\mathbf{G} \subset \mathscr{V}_{b,\text{Ker}\,\mathbf{C}}(\mathbf{A},\mathbf{B},\mathbf{C})$*;*
(3) $\mathscr{S}_{b,\text{Im}\,\mathbf{G}}(\mathbf{A},\mathbf{G},\mathbf{C}) \subset \mathscr{V}_{b,\text{Ker}\,\mathbf{C}}(\mathbf{A},\mathbf{B},\mathbf{C})$*;*
(4) $\mathscr{S}_{b,\text{Im}\,\mathbf{G}}(\mathbf{A},\mathbf{G},\mathbf{C}) \subset \text{Ker}\,\mathbf{C}$*;*
(5) The matrix triples $(\mathbf{A},\mathbf{B},\mathbf{C})$ *and* $(\mathbf{A},\mathbf{G},\mathbf{C})$ *have no invariant zeros on the imaginary axis.*

The geometric subspaces can be computed by virtue of the special coordinate basis proposed by Sannuti & Saberi (1987) (reviewed in Appendix A) using available software, either numerically (Liu et al., 2005) or symbolically (Grip & Saberi, 2010).

Definition 1 *For given $\beta > 0$ and integer $N_0 \geq 1$, $\mathscr{G}_\beta$ is the set of graphs composed of N nodes where $N \leq N_0$ such that every $\mathscr{G} \in \mathscr{G}_\beta$ has a directed spanning tree, and the eigenvalues of its Laplacian, denoted λ_i for $i = 1,\cdots,N$, satisfy* $\text{Re}\{\lambda_i\} > \beta$ *for $\lambda_i \neq 0$.* ◀

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A directed graph $\mathscr{G}$ has a directed spanning tree if it has a node from which there are directed paths to every other nodes. The Laplacian G associated with $\mathscr{G} \in \mathscr{G}_\beta$ has a simple eigenvalue at zero and the rest are located in $\mathbb{C}^+$, according to (Ren & Beard, 2005).

3.2 Protocol Development

To solve Problem 1, we introduce a distributed observer-based protocol parameterized in terms of a tuning parameter $\varepsilon \in (0,1]$ in the form of

$$\begin{cases} \dot{\hat{\mathbf{x}}}_i = (\mathbf{A} - \mathbf{B}\mathbf{F}_{\rm con}(\varepsilon))\hat{\mathbf{x}}_i + \mathbf{K}_{\rm obs}(\varepsilon)(\zeta_i - \hat{\zeta}_i) & \text{(5a)} \\ \mathbf{u}_i = -\mathbf{F}_{\rm con}(\varepsilon)\hat{\mathbf{x}}_i\,, \qquad i \in \mathfrak{S} & \text{(5b)} \end{cases}$$

where $\hat{\mathbf{x}}_i \in \mathbb{R}^n$; $\hat{\zeta}_i$ is given by (1c) where $\eta_i = \mathbf{C}\hat{\mathbf{x}}_i$. We present a step-by-step design procedure for determining the gains $\mathbf{F}_{\rm con}(\varepsilon)$ and $\mathbf{K}_{\rm obs}(\varepsilon)$. The algorithm makes use of the special coordinate basis for multivariable linear systems (Sannuti & Saberi, 1987); see Appendix A. The design procedure is given below.

Step 1 Find nonsingular transformations $\Gamma_{\rm x}$, $\Gamma_{\rm u}$ and $\Gamma_{\rm y}$ in order to represent the system characterized by the matrix triple $(\mathbf{A},\mathbf{B},\mathbf{C})$ into the SCB as stated in Appendix A. To that end, let

$$\mathbf{x}_i = \Gamma_{\rm x} x_i,\;\; \mathbf{y}_i = \Gamma_{\rm y}\,\mathrm{col}\{y_{{\rm d},i}, y_{{\rm b},i}\},\;\; \mathbf{u}_i = \Gamma_{\rm u}\,\mathrm{col}\{u_{{\rm d},i}, u_{{\rm c},i}\} \quad (6)$$

where $x_i = \mathrm{col}\{x^-_{{\rm a},i}, x^+_{{\rm a},i}, x_{{\rm b},i}, x_{{\rm c},i}, x_{{\rm d},i}\}$, $x_{{\rm d},i} = \mathrm{col}\{x_{j{\rm d},i}\}$, $y_{{\rm d},i} = \mathrm{col}\{y_{j{\rm d},i}\}$ and $u_{{\rm d},i} = \mathrm{col}\{u_{j{\rm d},i}\}$, $\forall j \in \Omega \triangleq \{1,\cdots,{\rm r}\}$; see Appendix A. It yields the SCB for agent $i \in \mathfrak{S}$:

$$\dot{x}^-_{{\rm a},i} = A^-_{\rm a} x^-_{{\rm a},i} + L^-_{\rm ad} y_{{\rm d},i} + L^-_{\rm ab} y_{{\rm b},i} + G^-_{\rm a}\mathbf{w}_i \quad \text{(7a)}$$
$$\dot{x}^+_{{\rm a},i} = A^+_{\rm a} x^+_{{\rm a},i} + L^+_{\rm ad} y_{{\rm d},i} + L^+_{\rm ab} y_{{\rm b},i} + G^+_{\rm a}\mathbf{w}_i \quad \text{(7b)}$$
$$\dot{x}_{{\rm b},i} = A_{\rm b} x_{{\rm b},i} + L_{\rm bd} y_{{\rm d},i} + G_{\rm b}\mathbf{w}_i \quad \text{(7c)}$$
$$\begin{aligned}\dot{x}_{{\rm c},i} &= A_{\rm c} x_{{\rm c},i} + L_{\rm cd} y_{{\rm d},i} + L_{\rm cb} y_{{\rm b},i} \\ &\quad + B_{\rm c}(u_{{\rm c},i} + E^-_{\rm ca} x^-_{{\rm a},i} + E^+_{\rm ca} x^+_{{\rm a},i}) + G_{\rm c}\mathbf{w}_i \end{aligned} \quad \text{(7d)}$$

and for each $j \in \Omega$, there are:

$$\dot{x}_{j{\rm d},i} = A_{j{\rm d}} x_{j{\rm d},i} + L_{j{\rm d}} y_{{\rm d},i} + B_{j{\rm d}}(u_{j{\rm d},i} + E_j x_i) + G_{j{\rm d}}\mathbf{w}_i \quad \text{(7e)}$$

In addition, $y_{j{\rm d},i} = C_{j{\rm d}} x_{j{\rm d},i}$, $y_{{\rm d},i} = C_{\rm d} x_{{\rm d},i}$, and $y_{{\rm b},i} = C_{\rm b} x_{{\rm b},i}$ where $C_{\rm d} = \mathrm{diag}\{C_{j{\rm d}}\}$ $\forall j \in \Omega$. It is also used that $\mathbf{G} = \Gamma_{\rm x}\,\mathrm{stack}\{G^-_{\rm a}, G^+_{\rm a}, G_{\rm b}, G_{\rm c}, G_{\rm d}\}$ where $G_{\rm d} = \mathrm{stack}\{G_{j{\rm d}}\}$, $\forall j \in \Omega$, where $G_{j{\rm d}} \in \mathbb{R}^{jq_j \times \omega}$. The dimensions of the variables as well as the size and the structure of the matrices conform with the SCB stated in Appendix A. Define the following matrices

$$A_{\rm s} = \begin{bmatrix} A^+_{\rm a} & L^+_{\rm ab} C_{\rm b} \\ 0 & A_{\rm b} \end{bmatrix}, \qquad L_{\rm sd} = \begin{bmatrix} L^+_{\rm ad} \\ L_{\rm bd} \end{bmatrix} \quad (8)$$

Step 2 Select the feedback gain matrices $F^+_{\rm a}$, $F_{\rm b}$, $F_{\rm c}$ and $F_{j{\rm d}}$ for $j = 1,\cdots,{\rm r}$ such that the following matrices become Hurwitz stable:

$$A_{\rm cc} = A_{\rm c} - B_{\rm c}F_{\rm c},\;\; A^*_{j{\rm d}} = A_{j{\rm d}} - B_{j{\rm d}}F_{j{\rm d}},\;\; A_{\rm ss} = A_{\rm s} - L_{\rm sd}F_{\rm s}$$

where $F_{\rm s} = [F^+_{\rm a},\, F_{\rm b}]$. Since the pairs $(A_{\rm c}, B_{\rm c})$ and $(A_{j{\rm d}}, B_{j{\rm d}})$ are controllable and the pair $(A_{\rm s}, L_{\rm sd})$ is stabilizable under Assumption 1-(1), the existence of $F^+_{\rm a}$, $F_{\rm b}$, $F_{\rm c}$ and $F_{j{\rm d}}$ is guaranteed. The dimensions of the gains $F^+_{\rm a}$, $F_{\rm b}$, $F_{\rm c}$ and $F_{j{\rm d}}$ are $p_{\rm d} \times n^+_{\rm a}$, $p_{\rm d} \times n_{\rm b}$, $m_{\rm c} \times n_{\rm c}$, and $q_j \times jq_j$, respectively.

Step 3 For every $j = 1,\cdots,{\rm r}$, define $\check{S}_j \in \mathbb{R}^{jq_j \times jq_j}$ as

$$\check{S}_j(\varepsilon) = \mathrm{diag}\{{\rm I}_{q_j}, \varepsilon {\rm I}_{q_j}, \cdots, \varepsilon^{j-1}{\rm I}_{q_j}\} \quad (9)$$

where $\varepsilon \in (0,1]$ is the tuning parameter and will be specified later. Also, for $j = 1,\cdots,{\rm r}$, define

$$F_{j{\rm d}\varepsilon} = \varepsilon^{-j} F_{j{\rm d}} \check{S}_j \qquad F_{{\rm d}\varepsilon} = \mathrm{diag}\{F_{j{\rm d}\varepsilon}\} \quad (10)$$

Step 4 Form $\mathscr{F}_{\rm c} \in \mathbb{R}^{m_{\rm c} \times n}$ and $\mathscr{F}_{\rm d} \in \mathbb{R}^{m_{\rm d} \times n}$ as below.

$$\mathscr{F}_{\rm c} = \begin{bmatrix} 0 & 0 & 0 & F_{\rm c} & 0 & \cdots & 0 \end{bmatrix}, \qquad \mathscr{F}_{\rm d} = \mathscr{F}_{\rm dd} + \mathscr{F}_{{\rm d}\varepsilon}$$

where $\mathscr{F}_{\rm dd} = \mathrm{stack}\{E_j\}$ for $j = 1,\cdots,{\rm r}$, and

$$\mathscr{F}_{{\rm d}\varepsilon} = \begin{bmatrix} 0 & F_{{\rm d}\varepsilon} C^{\rm T}_{\rm d} F^+_{\rm a} & F_{{\rm d}\varepsilon} C^{\rm T}_{\rm d} F_{\rm b} & 0 & F_{{\rm d}\varepsilon} \end{bmatrix}$$

Now, find $\mathbf{F}_{\rm con}(\varepsilon)$ as

$$\mathbf{F}_{\rm con}(\varepsilon) = \Gamma_{\rm u} \begin{bmatrix} \mathscr{F}_{\rm d} \\ \mathscr{F}_{\rm c} \end{bmatrix} \Gamma^{-1}_{\rm x} \quad (11)$$

Step 5 Find nonsingular transformations $\bar{\Gamma}_{\rm x}$, $\bar{\Gamma}_{\rm w}$ and $\bar{\Gamma}_{\rm y}$ in order to represent the system characterized by the matrix triple $(\mathbf{A},\mathbf{G},\mathbf{C})$ into the SCB as stated in Appendix A. For simplicity, we keep the notation used in Step 1 unchanged and place bars on the variables, matrices, and their dimensions. Then choose

$$\mathbf{x}_i = \bar{\Gamma}_{\rm x}\bar{x}_i, \qquad \mathbf{y}_i = \bar{\Gamma}_{\rm y}\begin{bmatrix} \bar{y}_{{\rm d},i} \\ \bar{y}_{{\rm b},i} \end{bmatrix}, \qquad \mathbf{w}_i = \bar{\Gamma}_{\rm w}\begin{bmatrix} w_{{\rm d},i} \\ w_{{\rm c},i} \end{bmatrix} \quad (12)$$

where $\bar{x}_i = \mathrm{col}\{\bar{x}^-_{{\rm a},i}, \bar{x}^+_{{\rm a},i}, \bar{x}_{{\rm b},i}, \bar{x}_{{\rm c},i}, \bar{x}_{{\rm d},i}\}$. It yields the SCB for agent $i \in \mathfrak{S}$:

$$\dot{\bar{x}}^-_{{\rm a},i} = \bar{A}^-_{\rm a}\bar{x}^-_{{\rm a},i} + \bar{L}^-_{\rm ad}\bar{y}_{{\rm d},i} + \bar{L}^-_{\rm ab}\bar{y}_{{\rm b},i} + \bar{B}^-_{\rm a}\mathbf{u}_i \quad \text{(13a)}$$
$$\dot{\bar{x}}^+_{{\rm a},i} = \bar{A}^+_{\rm a}\bar{x}^+_{{\rm a},i} + \bar{L}^+_{\rm ad}\bar{y}_{{\rm d},i} + \bar{L}^+_{\rm ab}\bar{y}_{{\rm b},i} + \bar{B}^+_{\rm a}\mathbf{u}_i \quad \text{(13b)}$$
$$\dot{\bar{x}}_{{\rm b},i} = \bar{A}_{\rm b}\bar{x}_{{\rm b},i} + \bar{L}_{\rm bd}\bar{y}_{{\rm d},i} + \bar{B}_{\rm b}\mathbf{u}_i \quad \text{(13c)}$$
$$\begin{aligned}\dot{\bar{x}}_{{\rm c},i} &= \bar{A}_{\rm c}\bar{x}_{{\rm c},i} + \bar{L}_{\rm cd}\bar{y}_{{\rm d},i} + \bar{L}_{\rm cb}\bar{y}_{{\rm b},i} \\ &\quad + \bar{G}_{\rm c}(w_{{\rm c},i} + \bar{E}^-_{\rm ca}\bar{x}^-_{{\rm a},i} + \bar{E}^+_{\rm ca}\bar{x}^+_{{\rm a},i}) + \bar{B}_{\rm c}\mathbf{u}_i \end{aligned} \quad \text{(13d)}$$

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and considering that $\bar{x}_{\mathrm{d},i} = \mathrm{col}\{\bar{x}_{j\mathrm{d},i}\}$, $\bar{y}_{\mathrm{d},i} = \mathrm{col}\{\bar{y}_{j\mathrm{d},i}\}$, and $w_{\mathrm{d},i} = \mathrm{col}\{w_{j\mathrm{d},i}\}$, $\forall j \in \bar{\Omega} \triangleq \{1, \cdots, \bar{r}\}$, there are:

$$\dot{\bar{x}}_{j\mathrm{d},i} = \bar{A}_{j\mathrm{d}}\bar{x}_{j\mathrm{d},i} + \bar{L}_{j\mathrm{d}}\bar{y}_{\mathrm{d},i} + \bar{G}_{j\mathrm{d}}(w_{j\mathrm{d},i} + \bar{E}_j\bar{x}_i) + \bar{B}_{j\mathrm{d}}\mathbf{u}_i \quad (13\mathrm{e})$$

Moreover, $\bar{y}_{j\mathrm{d},i} = \bar{C}_{j\mathrm{d}}\bar{x}_{j\mathrm{d},i}$, $\bar{y}_{\mathrm{d},i} = \bar{C}_{\mathrm{d}}\bar{x}_{\mathrm{d},i}$ where $\bar{C}_{\mathrm{d}} = \mathrm{diag}\{\bar{C}_{j\mathrm{d}}\}$, $\forall j \in \bar{\Omega}$, and $\bar{y}_{\mathrm{b},i} = \bar{C}_{\mathrm{b}}\bar{x}_{\mathrm{b},i}$. Also, we have used the following notation: $\mathbf{B} = \bar{\Gamma}_{\mathrm{x}}\ \mathrm{stack}\{\bar{B}_{\mathrm{a}}^-, \bar{B}_{\mathrm{a}}^+, \bar{B}_{\mathrm{b}}, \bar{B}_{\mathrm{c}}, \bar{B}_{\mathrm{d}}\}$ where $\bar{B}_{\mathrm{d}} = \mathrm{stack}\{\bar{B}_{j\mathrm{d}}\}$, $\forall j \in \bar{\Omega}$, where $\bar{B}_{j\mathrm{d}} \in \mathbb{R}^{j\bar{q}_j \times m}$. The dimensions of the variables and the size and the structure of the matrices as well as the matrix partitioning conform with the SCB stated in Appendix A. Define the following matrices

$$\bar{A}_{\mathrm{s}} = \begin{bmatrix} \bar{A}_{\mathrm{a}}^+ & 0 \\ \bar{G}_{\mathrm{c}}\bar{E}_{\mathrm{ca}}^+ & \bar{A}_{\mathrm{c}} \end{bmatrix}, \quad \bar{E}_{\mathrm{ds}} = \mathrm{stack}\{[\bar{E}_{j\mathrm{a}}^+, \quad \bar{E}_{j\mathrm{c}}]\} \quad (14)$$

Step 6 Select $\bar{K}_{\mathrm{a}}^+$ and $\bar{K}_{\mathrm{c}}$ such that

$$\bar{A}_{\mathrm{ss}} = \bar{A}_{\mathrm{s}} - \bar{K}_{\mathrm{sd}}\bar{E}_{\mathrm{ds}} \qquad \text{where} \qquad \bar{K}_{\mathrm{sd}} = \begin{bmatrix} \bar{K}_{\mathrm{a}}^+ \\ \bar{K}_{\mathrm{c}} \end{bmatrix}$$

is made Hurwitz stable. Such $\bar{K}_{\mathrm{sd}}$ exists under Assumption 1-(2), which implies the pair $(\bar{A}_{\mathrm{s}}, \bar{E}_{\mathrm{ds}})$ is detectable. Let $\tau \in (0, \beta]$, find $\bar{P}_{\mathrm{b}} = \bar{P}_{\mathrm{b}}^{\mathrm{T}} > 0$ and $\bar{P}_{j\mathrm{d}} = \bar{P}_{j\mathrm{d}}^{\mathrm{T}} > 0$ which solve the following algebraic Riccati equations:

$$\bar{A}_{\mathrm{b}}\bar{P}_{\mathrm{b}} + \bar{P}_{\mathrm{b}}\bar{A}_{\mathrm{b}}^{\mathrm{T}} - 2\tau\bar{P}_{\mathrm{b}}\bar{C}_{\mathrm{b}}^{\mathrm{T}}\bar{C}_{\mathrm{b}}\bar{P}_{\mathrm{b}} = -\mathrm{I}_{\bar{n}_{\mathrm{b}}}$$
$$\bar{A}_{j\mathrm{d}}\bar{P}_{j\mathrm{d}} + \bar{P}_{j\mathrm{d}}\bar{A}_{j\mathrm{d}}^{\mathrm{T}} - 2\tau\bar{P}_{j\mathrm{d}}\bar{C}_{j\mathrm{d}}^{\mathrm{T}}\bar{C}_{j\mathrm{d}}\bar{P}_{j\mathrm{d}} = -\mathrm{I}_{j\bar{q}_j}$$

for every $j \in \bar{\Omega}$. Then, define

$$\bar{K}_{\mathrm{b}} = \bar{P}_{\mathrm{b}}\bar{C}_{\mathrm{b}}^{\mathrm{T}}, \qquad \bar{K}_{j\mathrm{d}} = \bar{P}_{j\mathrm{d}}\bar{C}_{j\mathrm{d}}^{\mathrm{T}}, \qquad j \in \bar{\Omega}$$

The existence of such $\bar{P}_{\mathrm{b}}$ and $\bar{P}_{j\mathrm{d}}$ follows from the observability of the pairs $(\bar{C}_{\mathrm{b}}, \bar{A}_{\mathrm{b}})$ and $(\bar{C}_{j\mathrm{d}}, \bar{A}_{j\mathrm{d}})$. We point out that $\bar{K}_{\mathrm{a}}^+$, $\bar{K}_{\mathrm{b}}$, $\bar{K}_{\mathrm{c}}$, and $\bar{K}_{j\mathrm{d}}$ have the dimensions of $\bar{n}_{\mathrm{a}}^+ \times \bar{p}_{\mathrm{d}}$, $\bar{n}_{\mathrm{b}} \times \bar{p}_{\mathrm{b}}$, $\bar{n}_{\mathrm{c}} \times \bar{p}_{\mathrm{d}}$ and $j\bar{q}_j \times \bar{q}_j$, respectively.

Step 7 Define the matrix $\bar{S}_j \in \mathbb{R}^{j\bar{q}_j \times j\bar{q}_j}$, for every $j \in \bar{\Omega}$, as

$$\bar{S}_j(\tilde{\varepsilon}_j) = \mathrm{diag}\{\mathrm{I}_{\bar{q}_j}, \tilde{\varepsilon}_j\mathrm{I}_{\bar{q}_j}, \cdots, \tilde{\varepsilon}_j^{j-2}\mathrm{I}_{\bar{q}_j}, \tilde{\varepsilon}_j^{j-1}\mathrm{I}_{\bar{q}_j}\} \quad (15)$$

where $\tilde{\varepsilon}_j = \varepsilon^{\frac{\bar{r}}{j}}$. Also, for every $j \in \bar{\Omega}$, define

$$\bar{K}_{j\mathrm{d}\varepsilon} = \tilde{\varepsilon}_j^{-1}\bar{S}_j^{-1}\bar{K}_{j\mathrm{d}}, \qquad \bar{K}_{\mathrm{d}\varepsilon} = \mathrm{diag}\{\bar{K}_{j\mathrm{d}\varepsilon}\} \quad (16)$$

Step 8 Form $\mathscr{K}_{\mathrm{b}} \in \mathbb{R}^{n \times \bar{p}_{\mathrm{b}}}$ and $\mathscr{K}_{\mathrm{d}\varepsilon} \in \mathbb{R}^{n \times \bar{p}_{\mathrm{d}}}$ as below:

$$\mathscr{K}_{\mathrm{d}\varepsilon} = \mathrm{stack}\{0,\ \bar{K}_{\mathrm{a}}^+\bar{G}_{\mathrm{d}}^{\mathrm{T}}\bar{K}_{\mathrm{d}\varepsilon},\ 0,\ \bar{K}_{\mathrm{c}}\bar{G}_{\mathrm{d}}^{\mathrm{T}}\bar{K}_{\mathrm{d}\varepsilon},\ \bar{K}_{\mathrm{d}\varepsilon}\} \quad (17\mathrm{a})$$
$$\mathscr{K}_{\mathrm{b}} = \mathrm{stack}\{0,\ 0,\ \bar{K}_{\mathrm{b}},\ 0,\ 0\} \quad (17\mathrm{b})$$

Let $\bar{G}_{\mathrm{d}} = \mathrm{diag}\{\bar{G}_{j\mathrm{d}}\}$, $\forall j \in \bar{\Omega}$. Now, obtain $\mathbf{K}_{\mathrm{obs}}(\varepsilon)$ using

$$\mathbf{K}_{\mathrm{obs}}(\varepsilon) = \bar{\Gamma}_{\mathrm{x}}\begin{bmatrix} \mathscr{K}_{\mathrm{d}\varepsilon} & \mathscr{K}_{\mathrm{b}} \end{bmatrix}\bar{\Gamma}_{\mathrm{y}}^{-1} \quad (18)$$

Theorem 1 formalizes the result.

Theorem 1 *Under Assumption 1 and for the set $\mathscr{G}_\beta$, the parameterized protocol* (5)*, where $\mathbf{F}_{\mathrm{con}}(\varepsilon)$ is selected as in* (11) *and $\mathbf{K}_{\mathrm{obs}}(\varepsilon)$ is selected as in* (18)*, solves Problem 1. Precisely, the following hold*

(i) for any given $\beta > 0$, there exists an $\varepsilon_1^* \in (0,1]$ such that, for every $\varepsilon \in (0, \varepsilon_1^*]$, synchronization is accomplished in the absence of disturbance; i.e. $\forall \varepsilon \in (0, \varepsilon_1^*]$ when $w = 0$

$$\mathfrak{e}_{i,j} = y_i - y_j \to 0, \quad \forall i, j \in \mathfrak{S}, i > j \quad \text{as} \quad t \to \infty$$

(ii) for any given $\gamma > 0$, there exists an $\varepsilon_2^* \in (0, \varepsilon_1^*]$ such that for every $\varepsilon \in (0, \varepsilon_2^*]$, the closed-loop transfer function from w to $\mathfrak{e}$ satisfies $\|T_{w\mathfrak{e}}(s)\|_\infty < \gamma$.

4 Simulation Result

A homogeneous network of four non-introspective agents is considered as depicted in Fig. 1. Each agent is described by the following state-space model:

$$\dot{\mathbf{x}}_i = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\mathbf{x}_i + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\mathbf{u}_i + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\mathbf{w}_i, \ \mathbf{y}_i = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\mathbf{x}_i$$

We intend to solve the problem of H_∞ almost synchronization for this networked dynamical system. Clearly, each agent, which has one input and two outputs, satisfies Assumption 1. Notice that the system is already represented in the SCB with respect to both $\mathbf{B}$ and $\mathbf{G}$, and both the SCB are identical; each system is left-invertible, minimum phase, and it has one infinite zero of order one. Consequently, it is straightforward to verify that $\mathscr{S}_{b,\mathrm{Im}\,\mathbf{G}}(\mathbf{A}, \mathbf{G}, \mathbf{C})$ is empty. Thus, Assumption 1-(4) and (5) hold. Since $\mathrm{Im}\,\mathbf{G} \subset \mathscr{R}^*_{\mathrm{Ker}\,\mathbf{C}}(\mathbf{A}, \mathbf{B}, \mathbf{C})$, Assumption 1-(3) also holds. We assume that $\mathscr{G} \in \mathscr{G}_\beta$ with $\beta = 3.5$. Therefore, Theorem 1 ensures that the problem is solvable using the controller (5).

According to Fig. 1, the eigenvalues of the Laplacian of the communication network graph are $0, 7, 4 \pm j2.2361$. Taking $\tau = 3.5$, we found $\bar{K}_{\mathrm{d}} = 0.378$ and $\bar{K}_{\mathrm{b}} = 0.5469$. We also select $F_{\mathrm{b}} = 5$ and $F_{\mathrm{d}} = 20$. The protocol gains, $\mathbf{K}_{\mathrm{obs}}(\varepsilon)$ and $\mathbf{F}_{\mathrm{con}}(\varepsilon)$, are given by

$$\mathbf{K}_{\mathrm{obs}}(\varepsilon) = \begin{bmatrix} 0 & 0 \\ 0 & \bar{K}_{\mathrm{b}} \\ \frac{1}{\varepsilon}\bar{K}_{\mathrm{d}} & 0 \end{bmatrix}, \qquad \mathbf{F}_{\mathrm{con}}(\varepsilon) = \begin{bmatrix} 0 \\ \frac{1}{\varepsilon}F_{\mathrm{d}}F_{\mathrm{b}} \\ \frac{1}{\varepsilon}F_{\mathrm{d}} \end{bmatrix}^{\mathrm{T}}$$

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We assume that $\mathbf{w}_i(t) = i + \sin(\frac{i}{2}t + \frac{\pi}{i})$. The result of the simulation is shown in Fig. 2, where we have plotted $\zeta_{k,i} = \sum_{i=1}^{4} g_{ij}\mathbf{y}_{k,j}$ for $k = 1,2$. Clearly, $\zeta_{k,i} = 0$ means $\mathbf{y}_{k,i} = \mathbf{y}_{k,j}$. In Fig. 3, a comparison between $\varepsilon = 0.01$ and $\varepsilon = 0.03$ is presented. Reducing ε results in a more accurate synchronization.

Theorem 1 guarantees H_∞ almost synchronization for sufficiently small ε, which amplifies $\mathbf{F}_{\text{con}}(\varepsilon)$ and $\mathbf{K}_{\text{obs}}(\varepsilon)$. In other words, Theorem 1 provides lower bounds for $\mathbf{F}_{\text{con}}(\varepsilon)$ and $\mathbf{K}_{\text{obs}}(\varepsilon)$. However, for practical purposes where measurements are corrupted with noise, these gains should be tuned appropriately in order to ensure synchronization by choosing sufficiently high, and to achieve the best performance by limiting the magnitude of the gains.

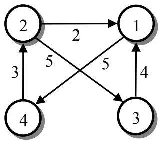

Fig. 1. The communication topology of the network.

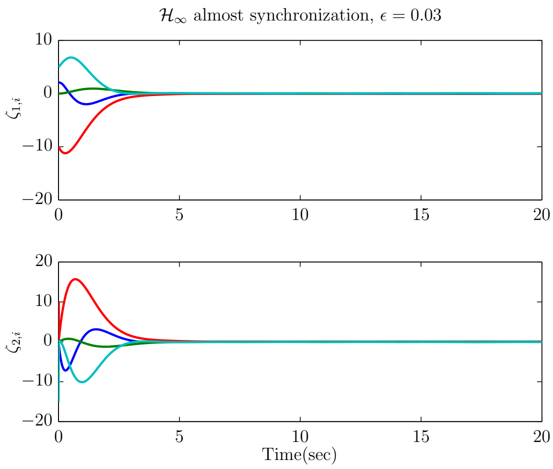

Fig. 2. H_∞ almost synchronization. The upper plot shows $\zeta_{1,i}$ and the lower plot shows $\zeta_{2,i}$.

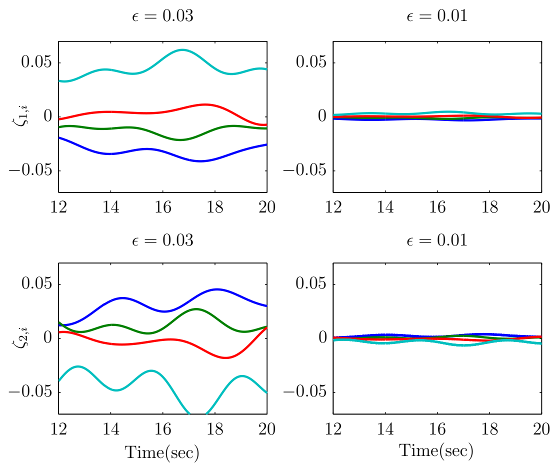

Fig. 3. H_∞ almost synchronization. A blow-up of the results for $\varepsilon = 0.01$ and $\varepsilon = 0.03$. The upper plots show $\zeta_{1,i}$ and the lower plots show $\zeta_{2,i}$.

5 Conclusion

We studied the problem of synchronization for multi-agent systems with identical linear dynamics under directional communication structures and in the presence of external disturbances. Utilizing the time-scale assignment technique and the geometric control theory, we proposed a family of dynamic protocols ensuring any accuracy of synchronization in the sense of the H_∞ norm of the closed-loop transfer function from disturbance to the synchronization error.

References

Bauso, D., Giarr, L., & Pesenti, R. (2009). Consensus for networks with unknown but bounded disturbances. *SIAM Journal on Control and Optimization*, *48*, 1756–1770.

Cao, Y., Yu, W., Ren, W., & Chen, G. (2013). An overview of recent progress in the study of distributed multi-agent coordination. *IEEE Transactions on Industrial Informatics*, *9*, 427–438.

Du, H., Li, S., & Shi, P. (2012). Robust consensus algorithm for second-order multi-agent systems with external disturbances. *International Journal of Control*, *85*, 1913–1928.

Fax, J. A., & Murray, R. M. (2004). Information flow and cooperative control of vehicle formations. *IEEE Transactions on Automatic Control*, *49*, 1465–1476.

Grip, H. F., & Saberi, A. (2010). Structural decomposition of linear multivariable systems using symbolic computations. *International Journal of Control*, *83*, 1414–1426.

Grip, H. F., Yang, T., Saberi, A., & Stoorvogel, A. A. (2012). An observer-based output synchronization protocol for heterogeneous networks of non-introspective agents. In *proc. American Control Conf.*. Montreal, Canada.

Hong-Yong, Y., Lei, G., & Chao, H. (2011). Robust consensus of multi-agent systems with uncertain exogenous disturbances. *Communications in Theoretical Physics*, *56*, 1161.

Jadbabaie, A., Lin, J., & Morse, A. (2003). Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Transactions on Automatic Control*, *48*, 988 – 1001.

Kokotovic, P. V., O'Reilly, J., & Khalil, H. K. (1986). *Singular Perturbation Methods in Control: Analysis and Design*. Orlando, FL, USA: Academic Press, Inc.

Li, S., Du, H., & Lin, X. (2011a). Finite-time consensus algorithm for multi-agent systems with double-integrator dynamics. *Automatica*, *47*, 1706 – 1712.

Li, T., & Zhang, J.-F. (2009). Mean square average-consensus under measurement noises and fixed topologies: Necessary and sufficient conditions. *Automatica*, *45*, 1929 – 1936.

Li, Z., Duan, Z., & Chen, G. (2011b). On H_∞ and H_2 performance regions of multi-agent systems. *Automatica*, *47*, 797 – 803.

Li, Z., Duan, Z., Chen, G., & Huang, L. (2010). Consensus of multiagent systems and synchronization of complex networks: A unified viewpoint. *IEEE Transactions on Circuits and Systems I: Regular Papers*, *57*, 213 –224.

Li, Z., Liu, X., Fu, M., & Xie, L. (2012). Global H_∞ consensus of multi-agent systems with lipschitz nonlinear dynamics. available at http://arxiv.org/abs/1202.5447.

Lin, P., & Jia, Y. (2010). Robust H_∞ consensus analysis of a class of second-order multi-agent systems with uncertainty. *Control Theory Applications, IET*, *4*, 487 –498.

Preprint submitted to Automatica
Received February 10, 2014 14:17:45 PST

Lin, P., Jia, Y., & Li, L. (2008). Distributed robust H_∞ consensus control in directed networks of agents with time-delay. *Systems & Control Letters*, *57*, 643 – 653.

Liu, X., Chen, B., & Lin, Z. (2005). Linear systems toolkit in matlab: structural decompositions and their applications. *Journal of Control Theory and Applications*, *3*, 287–294.

Liu, Y., & Jia, Y. (2010). Consensus problem of high-order multi-agent systems with external disturbances: An H_∞ analysis approach. *International Journal of Robust and Nonlinear Control*, *20*, 1579–1593.

Liu, Y., & Jia, Y. (2011). Robust H_∞ consensus control of uncertain multi-agent systems with time delays. *International Journal of Control, Automation and Systems*, *9*, 1086–1094.

Massioni, P., & Verhaegen, M. (2009). Distributed control for identical dynamically coupled systems: A decomposition approach. *IEEE Transactions on Automatic Control*, *54*, 124 –135.

Mo, L., & Jia, Y. (2011). H_∞ consensus control of a class of high-order multi-agent systems. *Control Theory Applications, IET*, *5*, 247 – 253.

Moreau, L. (2005). Stability of multiagent systems with time-dependent communication links. *IEEE Transactions on Automatic Control*, *50*, 169 – 182.

Olfati-Saber, R., Fax, J., & Murray, R. (2007). Consensus and cooperation in networked multi-agent systems. *Proceedings of the IEEE*, *95*, 215 –233.

Olfati-Saber, R., & Murray, R. (2004). Consensus problems in networks of agents with switching topology and time-delays. *IEEE Transactions on Automatic Control*, *49*, 1520 – 1533.

Ozcetin, H. K., Saberi, A., & Sannuti, P. (1992). Design for H_∞ almost disturbance decoupling problem with internal stability via state or measurement feedback–singular perturbation approach. *International Journal of Control*, *55*, 901–944.

Peymani, E., Grip, H. F., Saberi, A., Wang, X., & Fossen, T. I. (2014). H_∞ almost synchronization for heterogeneous networks of introspective agents under external disturbances. *Automatica*, *0*, 0 – 0. To appear.

Ren, W., & Beard, R. (2005). Consensus seeking in multiagent systems under dynamically changing interaction topologies. *IEEE Transactions on Automatic Control*, *50*, 655 – 661.

Ren, W., & Cao, Y. (2011). *Distributed Coordination of Multi-agent Networks*. Communications and Control Engineering, Springer-Verlag.

Saberi, A., Stoorvogel, A. A., & Sannuti, P. (2012). *Internal and External Stabilization of Linear Systems with Constraints*. Birkhuser Boston.

Sannuti, P., & Saberi, A. (1987). Special coordinate basis for multivariable linear-systems - finite and infinite zero structure, squaring down and decoupling. *International Journal of Control*, *45*, 1655–1704.

Shen, B., Wang, Z., & Hungpas, Y. (2010). Distributed H_∞-consensus filtering in sensor networks with multiple missing measurements: The finite-horizon case. *Automatica*, *46*, 1682 – 1688.

Shen, B., Wang, Z., & Liu, X. (2011). Bounded H_∞ synchronization and state estimation for discrete time-varying stochastic complex networks over a finite horizon. *IEEE Transactions on Neural Networks*, *22*, 145–157.

Tanner, H., Pappas, G., & Kumar, V. (2004). Leader-to-formation stability. *IEEE Transactions on Robotics and Automation*, *20*, 443 – 455.

Trentelman, H. L., Stoorvogel, A., & Hautus, M. (2001). *Control Theory for Linear Systems*. Springer.

Ugrinovskii, V. (2011). Distributed robust filtering with H_∞ consensus of estimates. *Automatica*, *47*, 1 – 13.

Wang, Z., Ding, D., Dong, H., & Shu, H. (2013). consensus control for multi-agent systems with missing measurements: The finite-horizon case. *Systems & Control Letters*, *62*, 827 – 836.

Weiland, S., & Willems, J. (1989). Almost disturbance decoupling with internal stability. *IEEE Transactions on Automatic Control*, *34*, 277 –286.

Wonham, W. M. (1985). *Linear multivariable control: a geometric approach*. Springer; 3rd ed. edition.

Wu, C. W. (2007). *Synchronization in complex networks of nonlinear dynamical systems*. World Scientific.

Wu, C. W., & Chua, L. (1995a). Application of graph theory to the synchronization in an array of coupled nonlinear oscillators. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, *42*, 494 –497.

Wu, C. W., & Chua, L. (1995b). Application of kronecker products to the analysis of systems with uniform linear coupling. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, *42*, 775 –778.

A Special Coordinate Basis

The protocol development relies extensively on a special coordinate basis (SCB) – see e.g. (Saberi et al., 2012, Chapter 3) – originally proposed by Sannuti & Saberi (1987). This section is devoted to recall the SCB of linear systems and its pertinent properties. Consider a linear, time-invariant system described by

$$\Sigma : \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \qquad \mathbf{y} = \mathbf{C}\mathbf{x} \tag{A.1}$$

where $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$ and $\mathbf{C} \in \mathbb{R}^{p\times n}$. Also, $\mathbf{x} \in \mathbb{R}^n$ is the state, $\mathbf{u} \in \mathbb{R}^m$ is the control, and $\mathbf{y} \in \mathbb{R}^p$ is the output. According to (Sannuti & Saberi, 1987), for any system Σ characterized by the matrix triple $(\mathbf{A},\mathbf{B},\mathbf{C})$, there exist

(i) unique coordinate-free non-negative integers n_a^-, n_a^+, n_b, n_c, n_d, $1 \le \mathrm{r} \le n$, and q_j, $j = 1,\cdots,\mathrm{r}$.
(ii) nonsingular state, output and input transformations Γ_x, Γ_y, and Γ_u as $\mathbf{x} = \Gamma_x\tilde{\mathbf{x}}$, $\mathbf{y} = \Gamma_y\tilde{\mathbf{y}}$ and $\mathbf{u} = \Gamma_u\tilde{\mathbf{u}}$ such that

$$\tilde{\mathbf{x}} = \mathrm{col}\{x_a^-, x_a^+, x_b, x_c, x_d\},\ \ \tilde{\mathbf{y}} = \mathrm{col}\{y_d, y_b\},\ \ \tilde{\mathbf{u}} = \mathrm{col}\{u_d, u_c\}$$

where the states x_a^-, x_a^+, x_b, x_c, x_d have dimensions n_a^-, n_a^+, n_b, n_c, and n_d, respectively. Also,

$$u_d, y_d \in \mathbb{R}^{m_d = p_d} \qquad u_c \in \mathbb{R}^{m_c} \qquad y_b \in \mathbb{R}^{p_b}$$

which implies $p = p_d + p_b$ and $m = m_d + m_c$. Moreover, x_d, u_d and y_d are partitioned as

$$x_d = \mathrm{col}\{x_{jd}\} \qquad y_d = \mathrm{col}\{y_{jd}\} \qquad u_d = \mathrm{col}\{u_{jd}\}$$

for $j = 1,\cdots,\mathrm{r}$. Here, $x_{jd} \in \mathbb{R}^{jq_j}$ and $u_{jd}, y_{jd} \in \mathbb{R}^{q_j}$. For every $j \in \{1,\cdots,\mathrm{r}\}$, define

$$A_{jd} = \begin{bmatrix} 0 & \mathrm{I}_{q_j(j-1)} \\ 0 & 0 \end{bmatrix}, \quad B_{jd} = \begin{bmatrix} 0 \\ \mathrm{I}_{q_j} \end{bmatrix}, \quad C_{jd} = \begin{bmatrix} \mathrm{I}_{q_j} & 0 \end{bmatrix}$$

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Clearly, $A_{1\mathrm{d}}=0$, $B_{1\mathrm{d}}=C_{1\mathrm{d}}=\mathrm{I}_{q_1}$. The transformations take Σ into the SCB described by the following set of equations:

$$\dot{x}_{\mathrm{a}}^- = A_{\mathrm{a}}^- x_{\mathrm{a}}^- + L_{\mathrm{ad}}^- y_{\mathrm{d}} + L_{\mathrm{ab}}^- y_{\mathrm{b}} \tag{A.2a}$$
$$\dot{x}_{\mathrm{a}}^+ = A_{\mathrm{a}}^+ x_{\mathrm{a}}^+ + L_{\mathrm{ad}}^+ y_{\mathrm{d}} + L_{\mathrm{ab}}^+ y_{\mathrm{b}} \tag{A.2b}$$
$$\dot{x}_{\mathrm{b}} = A_{\mathrm{b}} x_{\mathrm{b}} + L_{\mathrm{bd}} y_{\mathrm{d}} \tag{A.2c}$$
$$\dot{x}_{\mathrm{c}} = A_{\mathrm{c}} x_{\mathrm{c}} + L_{\mathrm{cd}} y_{\mathrm{d}} + L_{\mathrm{cb}} y_{\mathrm{b}} + B_{\mathrm{c}}(u_{\mathrm{c}} + E_{\mathrm{ca}}^- x_{\mathrm{a}}^- + E_{\mathrm{ca}}^+ x_{\mathrm{a}}^+) \tag{A.2d}$$

and for each $j=1,\cdots,\mathrm{r}$, there are:

$$\dot{x}_{j\mathrm{d}} = A_{j\mathrm{d}} x_{j\mathrm{d}} + L_{j\mathrm{d}} y_{\mathrm{d}} + B_{j\mathrm{d}}(u_{j\mathrm{d}} + E_j \widetilde{\mathbf{x}}) \tag{A.2e}$$

where $E_j \in \mathbb{R}^{q_j \times n}$ and is appropriately partitioned as

$$E_j = \begin{bmatrix} E_{j\mathrm{a}}^- & E_{j\mathrm{a}}^+ & E_{j\mathrm{b}} & E_{j\mathrm{c}} & E_{j\mathrm{d}} \end{bmatrix}, \quad E_{j\mathrm{d}} = \begin{bmatrix} E_{j1} & \cdots & E_{j\mathrm{r}} \end{bmatrix}$$

where $E_{jk} \in \mathbb{R}^{q_j \times k q_k}$ for $k=1,\cdots,\mathrm{r}$ such that $E_j\widetilde{\mathbf{x}} = E_{j\mathrm{a}}^- x_{\mathrm{a}}^- + E_{j\mathrm{a}}^+ x_{\mathrm{a}}^+ + E_{j\mathrm{b}} x_{\mathrm{b}} + E_{j\mathrm{c}} x_{\mathrm{c}} + E_{j\mathrm{d}} x_{\mathrm{d}}$. The outputs are given by

$$y_{j\mathrm{d}} = C_{j\mathrm{d}} x_{j\mathrm{d}}, \qquad y_{\mathrm{d}} = C_{\mathrm{d}} x_{\mathrm{d}}, \qquad y_{\mathrm{b}} = C_{\mathrm{b}} x_{\mathrm{b}} \tag{A.2f}$$

where $C_{\mathrm{d}} = \mathrm{diag}\{C_{j\mathrm{d}}\}$ for all $j \in \{1,\cdots,\mathrm{r}\}$. One may consider that $L_{1\mathrm{d}}=0$. The presented SCB explicitly reveals the system's finite and infinite zero structures and the invertibility properties. The invariant zeros of the system Σ are the eigenvalues of A_{a}^- and A_{a}^+. We presume that the eigenvalues of A_{a}^- are located in $\mathbb{C}^-$ and the eigenvalues of A_{a}^+ are located in $\mathbb{C}^+$, assuming that the system has no invariant zeros on the imaginary axis. Thus, the system is non-minimum phase if x_{a}^+ is existent.

The $x_{j\mathrm{d}}$ subsystems show the infinite zero structure of the system. Thus, Σ has jq_j infinite zeros of order j. The subsystems x_{b} and x_{c} describe the invertibility properties of Σ. The reader should refer to (Sannuti & Saberi, 1987; Saberi et al., 2012) for details.

Clearly, $(C_{\mathrm{b}},A_{\mathrm{b}})$ and $(C_{j\mathrm{d}},A_{j\mathrm{d}})$ form observable pairs. In fact, the system Σ is observable (detectable) if and only if the pair $(C_{\mathrm{obs}},A_{\mathrm{obs}})$ is observable (detectable), where

$$C_{\mathrm{obs}} = \begin{bmatrix} E_{\mathrm{da}}^- & E_{\mathrm{da}}^+ & E_{\mathrm{dc}} \end{bmatrix}, \quad A_{\mathrm{obs}} = \begin{bmatrix} A_{\mathrm{a}}^- & 0 & 0 \\ 0 & A_{\mathrm{a}}^+ & 0 \\ B_{\mathrm{c}}E_{\mathrm{ca}}^- & B_{\mathrm{c}}E_{\mathrm{ca}}^+ & A_{\mathrm{c}} \end{bmatrix}$$

in which for $j=1,\cdots,\mathrm{r}$

$$E_{\mathrm{da}}^- = \mathrm{col}\{E_{j\mathrm{a}}^-\}, \quad E_{\mathrm{da}}^+ = \mathrm{col}\{E_{j\mathrm{a}}^+\}, \quad E_{\mathrm{dc}} = \mathrm{col}\{E_{j\mathrm{c}}\}$$

Moreover, $(A_{\mathrm{c}},B_{\mathrm{c}})$ and $(A_{j\mathrm{d}},B_{j\mathrm{d}})$ form controllable pairs. The system Σ is then controllable (stabilizable) if and only if the pair $(A_{\mathrm{con}},B_{\mathrm{con}})$ is controllable (stabilizable), where

$$A_{\mathrm{con}} = \begin{bmatrix} A_{\mathrm{a}}^- & 0 & L_{\mathrm{ab}}^- C_{\mathrm{b}} \\ 0 & A_{\mathrm{a}}^+ & L_{\mathrm{ab}}^+ C_{\mathrm{b}} \\ 0 & 0 & A_{\mathrm{b}} \end{bmatrix} \qquad B_{\mathrm{con}} = \begin{bmatrix} L_{\mathrm{ad}}^- \\ L_{\mathrm{ad}}^+ \\ L_{\mathrm{bd}} \end{bmatrix}$$

The geometric subspaces can be expressed in terms of appropriate unions of subspaces that describe the SCB of Σ. According to (Ozcetin et al., 1992), we have the following property which establishes a connection between the SCB and the geometric subspaces.

Property 1 *Suppose the state space is described by* $x_{\mathrm{a}}^- \oplus x_{\mathrm{a}}^+ \oplus x_{\mathrm{b}} \oplus x_{\mathrm{c}} \oplus x_{\mathrm{d}}$.

- $x_{\mathrm{a}}^- \oplus x_{\mathrm{c}} \oplus x_{\mathrm{d}}$ *spans* $\mathscr{V}_{b,\mathrm{Ker}\,\mathbf{C}}$;
- $x_{\mathrm{a}}^+ \oplus x_{\mathrm{c}}$ *spans* $\mathscr{S}_{b,\mathrm{Im}\,\mathbf{B}}$.

B Proof: Theorem 1

Estimation Error Dynamics for Agent 'i': We start the proof by finding the estimation error dynamics for agent $i \in \mathfrak{S}$. Define the estimation error as $\tilde{\mathbf{x}}_i = \mathbf{x}_i - \hat{\mathbf{x}}_i$, and find the dynamics according to (1) and (5). It gives rise to

$$\dot{\tilde{\mathbf{x}}}_i = \mathbf{A}\tilde{\mathbf{x}}_i + \mathbf{G}\mathbf{w}_i - \mathbf{K}_{\mathrm{obs}}(\varepsilon)(\zeta_i - \hat{\zeta}_i) \tag{B.1}$$

where $\zeta_i - \hat{\zeta}_i = \sum_{j=1}^N g_{ij}\mathbf{C}\tilde{\mathbf{x}}_j$ and $\mathbf{K}_{\mathrm{obs}}(\varepsilon)$, which is given by (18), is found using the coordinates corresponding to the SCB with respect to the triple $(\mathbf{A},\mathbf{G},\mathbf{C})$. Thus, using the transformation matrices found in Step 5 (see Eq. (12)), we transform (B.1) into that SCB. Let

$$\tilde{\mathbf{x}}_i = \bar{\Gamma}_{\mathrm{x}}\tilde{x}_i, \qquad \mathbf{C}\tilde{\mathbf{x}}_i = \bar{\Gamma}_{\mathrm{y}}\begin{bmatrix} \tilde{y}_{\mathrm{d},i} \\ \tilde{y}_{\mathrm{b},i} \end{bmatrix}, \qquad \mathbf{w}_i = \bar{\Gamma}_{\mathrm{w}}\begin{bmatrix} w_{\mathrm{d},i} \\ w_{\mathrm{c},i} \end{bmatrix}$$

where $\tilde{x}_i = \mathrm{col}\{\tilde{x}_{\mathrm{a},i}^-, \tilde{x}_{\mathrm{a},i}^+, \tilde{x}_{\mathrm{b},i}, \tilde{x}_{\mathrm{c},i}, \tilde{x}_{\mathrm{d},i}\}$ in which $\tilde{x}_{\mathrm{d},i} = \mathrm{col}\{\tilde{x}_{j\mathrm{d},i}\}$, $\tilde{y}_{\mathrm{d},i} = \mathrm{col}\{\tilde{y}_{j\mathrm{d},i}\}$, and $w_{\mathrm{d},i} = \mathrm{col}\{w_{j\mathrm{d},i}\}$ for all $j \in \bar{\Omega} = \{1,\cdots,\bar{\mathrm{r}}\}$. The dimensions conform with Appendix A, but we place bars on the variables; for example, $\tilde{x}_{j\mathrm{d},i} \in \mathbb{R}^{j\bar{q}_j}$ and $w_{j\mathrm{d},i}, \tilde{y}_{j\mathrm{d},i} \in \mathbb{R}^{\bar{q}_j}$. It is observed that

$$\tilde{\zeta}_{\mathrm{b},i} = \sum_{j=1}^N g_{ij}\tilde{y}_{\mathrm{b},j}, \qquad \tilde{\zeta}_{j\mathrm{d},i} = \sum_{k=1}^N g_{ik}\tilde{y}_{j\mathrm{d},k}$$

and $\tilde{\zeta}_{\mathrm{d},i} = \mathrm{col}\{\tilde{\zeta}_{j\mathrm{d},i}\}$ for $j \in \bar{\Omega}$. Then, in view of the SCB given by (13) along with (16) and (17), one can write

$$\dot{\tilde{x}}_{\mathrm{a},i}^- = \bar{A}_{\mathrm{a}}^-\tilde{x}_{\mathrm{a},i}^- + \bar{L}_{\mathrm{ad}}^-\tilde{y}_{\mathrm{d},i} + \bar{L}_{\mathrm{ab}}^-\tilde{y}_{\mathrm{b},i}$$
$$\dot{\tilde{x}}_{\mathrm{a},i}^+ = \bar{A}_{\mathrm{a}}^+\tilde{x}_{\mathrm{a},i}^+ + \bar{L}_{\mathrm{ad}}^+\tilde{y}_{\mathrm{d},i} + \bar{L}_{\mathrm{ab}}^+\tilde{y}_{\mathrm{b},i} - \bar{K}_{\mathrm{a}}^+\bar{G}_{\mathrm{d}}^{\mathrm{T}}\bar{K}_{\mathrm{d}\varepsilon}\tilde{\zeta}_{\mathrm{d},i}$$
$$\dot{\tilde{x}}_{\mathrm{b},i} = \bar{A}_{\mathrm{b}}\tilde{x}_{\mathrm{b},i} + \bar{L}_{\mathrm{bd}}\tilde{y}_{\mathrm{d},i} - \bar{K}_{\mathrm{b}}\tilde{\zeta}_{\mathrm{b},i}$$

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$$\dot{\tilde{x}}_{c,i} = \bar{A}_c\tilde{x}_{c,i} + \bar{L}_{cd}\tilde{y}_{d,i} + \bar{L}_{cb}\tilde{y}_{b,i} + \bar{G}_c(w_{c,i} + \bar{E}^-_{ca}\tilde{x}^-_{a,i} + \bar{E}^+_{ca}\tilde{x}^+_{a,i}) - \bar{K}_c\bar{G}^T_d\bar{K}_{d\varepsilon}\tilde{\zeta}_{d,i}$$
$$\dot{\tilde{x}}_{jd,i} = \bar{A}_{jd}\tilde{x}_{jd,i} + \bar{L}_{jd}\tilde{y}_{d,i} + \bar{G}_{jd}(w_{jd,i} + \bar{E}_j\tilde{x}_i) - \bar{K}_{jd\varepsilon}\tilde{\zeta}_{jd,i}$$
$$\tilde{y}_{jd,i} = \bar{C}_{jd}\tilde{x}_{jd,i}, \quad \tilde{y}_{d,i} = \bar{C}_d\tilde{x}_{d,i}, \quad \tilde{y}_{b,i} = \bar{C}_b\tilde{x}_{b,i}$$

The structure of matrices follows from the SCB, explained in Appendix A. For $j \in \bar{\Omega}$, define

$$\bar{E}^-_{da} = \text{stack}\{\bar{E}^-_{ja}\}, \bar{E}_{db} = \text{stack}\{\bar{E}_{jb}\}, \bar{L}_{dd} = \text{stack}\{\bar{L}_{jd}\}$$
$$\bar{L}_{sb} = \begin{bmatrix} \bar{L}^+_{ab} \\ \bar{L}_{cb} \end{bmatrix}, \quad \bar{L}_{sd} = \begin{bmatrix} \bar{L}^+_{ad} \\ \bar{L}_{cd} \end{bmatrix}, \quad \bar{G}_s = \begin{bmatrix} 0 \\ \bar{G}_c \end{bmatrix}$$

Define $\tilde{x}_{s,i} = \text{col}\{\tilde{x}^+_{a,i}, \tilde{x}_{c,i}\}$, and

$$\tilde{z}_{s,i} = \tilde{x}_{s,i} - \bar{K}_{sd}\bar{G}^T_d\tilde{x}_{d,i} \tag{B.2}$$

where $\bar{G}_d = \text{diag}\{\bar{G}_{jd}\}$ for $j \in \bar{\Omega}$. In view of Step 6, one may find $\dot{\tilde{z}}_{s,i} = \bar{A}_{ss}\tilde{z}_{s,i} + \bar{E}^-_{sa}\tilde{x}^-_{a,i} + \bar{E}_{sb}\tilde{x}_{b,i} + \bar{E}_{sd}\tilde{x}_{d,i} + \bar{G}_{ss}\mathbf{w}_i$ where $\bar{G}_{ss} = [-\bar{K}_{sd}, \; \bar{G}_s]\bar{\Gamma}^{-1}_w$ and

$$\bar{E}^-_{sa} = \bar{G}_s\bar{E}^-_{ca} - \bar{K}_{sd}\bar{E}^-_{da}, \qquad \bar{E}_{sb} = \bar{L}_{sb}\bar{C}_b - \bar{K}_{sd}\bar{E}_{db}$$
$$\bar{E}_{sd} = \bar{A}_{ss}\bar{K}_{sd}\bar{G}^T_d + \bar{L}_{sd}\bar{C}_d - \bar{K}_{sd}\bar{G}^T_d\bar{L}_{dd}\bar{C}_d - \bar{K}_{sd}\bar{E}_{dd}$$

Recalling the scaling matrix $\bar{S}_j$, in (15), we define $\widetilde{S} = \text{diag}\{\bar{S}_j\}$ for all $j \in \bar{\Omega}$. Considering the matrix $\bar{E}_{sd}$, one may demonstrate that $\varepsilon^{\bar{r}}\bar{E}_{sd}\widetilde{S}^{-1} = \varepsilon\mathbb{E}_{sd\varepsilon}$, indicating that $\|\varepsilon\mathbb{E}_{sd\varepsilon}\| = \mathscr{O}(\varepsilon)$. To show that, we partition $\bar{E}_{sd}$ as $\bar{E}_{sd} = [\bar{E}_{s1}, \bar{E}_{s2}, \cdots, \bar{E}_{s\bar{r}}]$ where $\bar{E}_{sk} \in \mathbb{R}^{(\bar{n}^+_a + \bar{n}_c)\times k\bar{q}_k}$ for every $k \in \bar{\Omega}$; therefore, it is obtained

$$\varepsilon^{\bar{r}}\bar{E}_{sd}\widetilde{S}^{-1} = \varepsilon^{\bar{r}}\left[\, \bar{E}_{s1}\bar{S}^{-1}_1 \;\cdots\; \bar{E}_{s\bar{r}}\bar{S}^{-1}_{\bar{r}} \,\right]$$

Since $\|\varepsilon^{\bar{r}}\bar{E}_{sk}\bar{S}^{-1}_k\| = \mathscr{O}(\varepsilon^{\bar{r}}\tilde{\varepsilon}^{-(k-1)}_k) = \mathscr{O}(\tilde{\varepsilon}_k)$, one may write $\varepsilon^{\bar{r}}\bar{E}_{sk}\bar{S}^{-1}_k = \varepsilon\bar{E}_{sk\varepsilon}$ for some appropriate $\bar{E}_{sk\varepsilon}$ which is uniformly bounded for all $\varepsilon \in (0,1]$. Therefore, $\mathbb{E}_{sd\varepsilon} = [\bar{E}_{s1\varepsilon}, \bar{E}_{s2\varepsilon}, \cdots, \bar{E}_{s\bar{r}\varepsilon}]$.

Denote $\bar{E}^*_{jd} = \bar{E}_{js}\bar{K}_{sd}\bar{G}^T_d + \bar{E}_{jd}$. Then, for every $j \in \bar{\Omega}$, one may show that

$$(\tilde{\varepsilon}_j\bar{S}_j\bar{L}_{jd}\bar{C}_d + \varepsilon^{\bar{r}}\bar{G}_{jd}\bar{E}^*_{jd})\widetilde{S}^{-1} = \varepsilon\bar{E}_{jd\varepsilon}$$

for some $\bar{E}_{jd\varepsilon}$ which is uniformly bounded for all $\varepsilon \in (0,1]$, and $\|\varepsilon\bar{E}_{jd\varepsilon}\| = \mathscr{O}(\varepsilon)$. Note that $\|\tilde{\varepsilon}_j\bar{S}_j\bar{L}_{jd}\bar{C}_d\widetilde{S}^{-1}\| = \mathscr{O}(\tilde{\varepsilon}_j)$.

Consider the following state transformations

$$\tilde{z}_{s\varepsilon,i} = \varepsilon^{\bar{r}}\tilde{z}_{s,i}, \qquad \tilde{x}_{jd\varepsilon,i} = \bar{S}_j\tilde{x}_{jd,i} \tag{B.3}$$

and define $\tilde{x}_{d\varepsilon,i} = \text{col}\{\tilde{x}_{jd\varepsilon,i}\}$ for all $j \in \bar{\Omega}$. Let $\bar{E}_{js} = [\bar{E}^+_{ja}, \bar{E}_{jc}]$. Consequently, the dynamics of the observation error system are given by:

$$\dot{\tilde{x}}^-_{a,i} = \bar{A}^-_a\tilde{x}^-_{a,i} + \bar{L}^-_{ad}\bar{C}_d\tilde{x}_{d\varepsilon,i} + \bar{L}^-_{ab}\tilde{y}_{b,i} \tag{B.4a}$$
$$\dot{\tilde{x}}_{b,i} = \bar{A}_b\tilde{x}_{b,i} + \bar{L}_{bd}\bar{C}_d\tilde{x}_{d\varepsilon,i} - \bar{K}_b\tilde{\zeta}_{b,i} \tag{B.4b}$$
$$\dot{\tilde{z}}_{s\varepsilon,i} = \bar{A}_{ss}\tilde{z}_{s\varepsilon,i} + \varepsilon^{\bar{r}}\bar{E}^-_{sa}\tilde{x}^-_{a,i} + \varepsilon^{\bar{r}}\bar{E}_{sb}\tilde{x}_{b,i} + \varepsilon\mathbb{E}_{sd\varepsilon}\tilde{x}_{d\varepsilon,i} + \varepsilon^{\bar{r}}\bar{G}_{ss}\mathbf{w}_i \tag{B.4c}$$
$$\mathbb{S}\dot{\tilde{x}}_{d\varepsilon,i} = \mathbb{A}_{dd}\tilde{x}_{d\varepsilon,i} + \varepsilon\mathbb{E}_{dd\varepsilon}\tilde{x}_{d\varepsilon,i} + \varepsilon^{\bar{r}}\mathbb{E}^-_{da}\tilde{x}^-_{a,i} + \varepsilon^{\bar{r}}\mathbb{E}_{db}\tilde{x}_{b,i} + \mathbb{E}_{ds}\tilde{z}_{s\varepsilon,i} + \varepsilon^{\bar{r}}\mathbb{G}_{dd}\mathbf{w}_i - \mathbb{K}_{dd}\tilde{\zeta}_{d,i} \tag{B.4d}$$

where we have used the following notations, for all $j \in \bar{\Omega}$,

$$\mathbb{A}_{dd} = \text{diag}\{\bar{A}_{jd}\} \qquad \mathbb{E}_{dd\varepsilon} = \text{stack}\{\bar{E}_{jd\varepsilon}\}$$
$$\mathbb{E}^-_{da} = \bar{G}_d\bar{E}^-_{da} \qquad \mathbb{E}_{db} = \bar{G}_d\bar{E}_{db}$$
$$\mathbb{E}_{ds} = \bar{G}_d\,\text{stack}\{\bar{E}_{js}\} \qquad \mathbb{G}_{dd} = [\bar{G}_d, \, 0]\bar{\Gamma}^{-1}_w$$
$$\mathbb{S} = \text{diag}\{\tilde{\varepsilon}_j I_{j\bar{q}_j}\} \qquad \mathbb{K}_{dd} = \text{diag}\{\bar{K}_{jd}\}$$

Note that $\tilde{\zeta}_{d,i} = \sum^N_{k=1} g_{ik}\bar{C}_d\tilde{x}_{d\varepsilon,k}$.

Dynamics of Agent 'i' under Feedback: We obtain the dynamics of agent i, (1), under the feedback (5b). Clearly, $\mathbf{u}_i = -\mathbf{F}_{con}(\varepsilon)\hat{\mathbf{x}}_i = -\mathbf{F}_{con}(\varepsilon)(\mathbf{x}_i - \tilde{\mathbf{x}}_i)$. Indeed, we obtain

$$\dot{\mathbf{x}}_i = (\mathbf{A} - \mathbf{B}\mathbf{F}_{con}(\varepsilon))\mathbf{x}_i + \mathbf{B}\mathbf{F}_{con}(\varepsilon)\tilde{\mathbf{x}}_i + \mathbf{G}\mathbf{w}_i$$
$$= (\mathbf{A} - \mathbf{B}\mathbf{F}_{con}(\varepsilon))\mathbf{x}_i + \mathbf{B}\Gamma_u\begin{bmatrix} \mathscr{F}_d \\ \mathscr{F}_c \end{bmatrix}\Gamma^{-1}_x\tilde{\mathbf{x}}_i + \mathbf{G}\mathbf{w}_i$$

The state-feedback gain $\mathbf{F}_{con}(\varepsilon)$ is calculated using the coordinates corresponding to the SCB with respect to the triple $(\mathbf{A}, \mathbf{B}, \mathbf{C})$. Therefore, it makes sense to transform the equations into that coordinate. In Step 1, using (6), we found (7). Thus, we intend to express $u_{c,i}$ and $u_{d,i}$ in terms of the coordinates corresponding to the SCB with respect to the triple $(\mathbf{A}, \mathbf{B}, \mathbf{C})$, and apply them to system equations (7).

Representing $\tilde{\mathbf{x}}_i$ in terms of the coordinates of the SCB with respect to the triple $(\mathbf{A}, \mathbf{B}, \mathbf{C})$, one may obtain $\tilde{\mathbf{x}}_i = \Gamma_x\check{\tilde{x}}_i$ where $\check{\tilde{x}}_i = \text{col}\{\check{\tilde{x}}^-_{a,i}, \check{\tilde{x}}^+_{a,i}, \check{\tilde{x}}_{b,i}, \check{\tilde{x}}_{c,i}, \check{\tilde{x}}_{d,i}\}$ in which $\check{\tilde{x}}_{d,i} = \text{col}\{\check{\tilde{x}}_{jd,i}\}$, $\forall j \in \Omega = \{1, \cdots, r\}$, where $\check{\tilde{x}}_{jd,i} \in \mathbb{R}^{jq_j}$. It clarifies that there exists a relation between estimation errors expressed in these two SCB, which is given by $\check{\tilde{x}}_i = \Gamma^{-1}_x\bar{\Gamma}_x\tilde{x}_i$.

That is, the components of one can be expressed as a linear combination of the other's components. According to Property 1, one can show

- $\check{\tilde{x}}^-_{a,i} \oplus \check{\tilde{x}}_{c,i} \oplus \check{\tilde{x}}_{d,i}$ spans $\mathscr{V}_{b,\text{Ker}\mathbf{C}}(\mathbf{A}, \mathbf{B}, \mathbf{C})$;
- $\tilde{x}^+_{a,i} \oplus \tilde{x}_{c,i}$ spans $\mathscr{S}_{b,\text{Im}\mathbf{G}}(\mathbf{A}, \mathbf{G}, \mathbf{C})$.

In accordance with Assumptions 1-(4),(5), we obtain

$$(\tilde{x}^+_{a,i} \oplus \tilde{x}_{c,i}) \subset (\check{\tilde{x}}^-_{a,i} \oplus \check{\tilde{x}}_{c,i} \oplus \check{\tilde{x}}_{d,i}),$$
$$(\tilde{x}^+_{a,i} \oplus \tilde{x}_{c,i}) \subset \text{Ker}\,\mathbf{C}$$

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The computation of the orthogonal complements of the above sub-spaces results in

$$(\check{\tilde{x}}_{\mathrm{a},i}^{+} \oplus \check{\tilde{x}}_{\mathrm{b},i}) \subset (\tilde{x}_{\mathrm{a},i}^{-} \oplus \tilde{x}_{\mathrm{b},i} \oplus \tilde{x}_{\mathrm{d},i}),$$
$$(\mathrm{Ker}\,\mathbf{C})^{\perp} \subset (\tilde{x}_{\mathrm{a},i}^{-} \oplus \tilde{x}_{\mathrm{b},i} \oplus \tilde{x}_{\mathrm{d},i})$$

It is observed that $C_{j\mathrm{d}}\check{\tilde{x}}_{j\mathrm{d},i} = \check{\tilde{x}}_{j1\mathrm{d},i} \in (\mathrm{Ker}\,\mathbf{C})^{\perp}$ for all $j \in \Omega$. Therefore, for all $j \in \Omega$ and every $i \in \mathfrak{S}$, one concludes that

- Assumptions 1-(4),(5) imply that $\check{\tilde{x}}_{j1\mathrm{d},i}$, $\check{\tilde{x}}_{\mathrm{a},i}^{+}$ and $\check{\tilde{x}}_{\mathrm{b},i}$ are expressed in terms of $\tilde{x}_{\mathrm{a},i}^{-}$, $\tilde{x}_{\mathrm{b},i}$ and $\tilde{x}_{\mathrm{d},i}$.
- Assumption 1-(3) implies that $G_{\mathrm{a}}^{+} = G_{\mathrm{b}} = 0$ since $\mathrm{Im}\,\mathbf{G}$ is spanned by $x_{\mathrm{a},i}^{-}$, $x_{\mathrm{c},i}$, and $x_{\mathrm{d},i}$.

Partition F_{a}^{+} and F_{b} such that $F_{\mathrm{a}}^{+} = \mathrm{stack}\{F_{j\mathrm{a}}^{+}\}$ and $F_{\mathrm{b}} = \mathrm{stack}\{F_{j\mathrm{b}}\}$ for all $j \in \Omega$, where $F_{j\mathrm{a}}^{+} \in \mathbb{R}^{q_j \times n_{\mathrm{a}}^{+}}$ and $F_{j\mathrm{b}} \in \mathbb{R}^{q_j \times n_{\mathrm{b}}}$. Define $F_{j\mathrm{s}} = [F_{j\mathrm{a}}^{+},\ F_{j\mathrm{b}}]$; thus, $F_{\mathrm{s}} = \mathrm{stack}\{F_{j\mathrm{s}}\}$ for all $j \in \Omega$. Denote $x_{\mathrm{s},i} = \mathrm{col}\{x_{\mathrm{a},i}^{+}, x_{\mathrm{b},i}\}$.

Define $s_{j\mathrm{d},i} = F_{j\mathrm{s}}x_{\mathrm{s},i} + y_{j\mathrm{d},i}$ and $z_{j\mathrm{d},i} = C_{j\mathrm{d}}^{\mathrm{T}}F_{j\mathrm{s}}x_{\mathrm{s},i} + x_{j\mathrm{d},i}$. Let $s_{\mathrm{d},i} = \mathrm{col}\{s_{j\mathrm{d},i}\}$ and $z_{\mathrm{d},i} = \mathrm{col}\{z_{j\mathrm{d},i}\}$ for all $j \in \Omega$. It implies that $s_{j\mathrm{d},i} = C_{j\mathrm{d}}z_{j\mathrm{d},i}$ and $s_{\mathrm{d},i} = F_{\mathrm{s}}x_{\mathrm{s},i} + y_{\mathrm{d},i}$. According to (8), $\dot{x}_{\mathrm{s},i} = A_{\mathrm{ss}}x_{\mathrm{s},i} + L_{\mathrm{sd}}s_{\mathrm{d},i}$. In light of the geometric assumptions and the state-feedback gain (11), we obtain

$$u_{\mathrm{c},i} = -F_{\mathrm{c}}(x_{\mathrm{c},i} - \check{\tilde{x}}_{\mathrm{c},i}) \tag{B.5a}$$
$$u_{j\mathrm{d},i} = -E_j x_i - F_{j\mathrm{d}\varepsilon}z_{j\mathrm{d},i} + \widetilde{u}_{j1,i} + \widetilde{u}_{j2,i} \tag{B.5b}$$

where $\widetilde{u}_{j1,i} = E_{j\mathrm{a}}^{-}\check{\tilde{x}}_{\mathrm{a},i}^{-} + E_{j\mathrm{a}}^{+}\check{\tilde{x}}_{\mathrm{a},i}^{+} + E_{j\mathrm{b}}\check{\tilde{x}}_{\mathrm{b},i} + E_{j\mathrm{c}}\check{\tilde{x}}_{\mathrm{c},i} + E_{j\mathrm{d}}\check{\tilde{x}}_{k\mathrm{d},i}$ and $\widetilde{u}_{j2,i} = F_{j\mathrm{d}\varepsilon}(C_{j\mathrm{d}}^{\mathrm{T}}F_{j\mathrm{a}}^{+}\check{\tilde{x}}_{\mathrm{a},i}^{+} + C_{j\mathrm{d}}^{\mathrm{T}}F_{j\mathrm{b}}\check{\tilde{x}}_{\mathrm{b},i} + \check{\tilde{x}}_{j\mathrm{d},i})$. From the assumptions on the geometric subspaces, it follows that $B_{\mathrm{c}}F_{\mathrm{c}}\check{\tilde{x}}_{\mathrm{c},i}$ is a linear combination of the components of $\tilde{x}_i$; i.e.

$$B_{\mathrm{c}}F_{\mathrm{c}}\check{\tilde{x}}_{\mathrm{c},i} = M_{\mathrm{ca}}^{-}\tilde{x}_{\mathrm{a},i}^{-} + M_{\mathrm{cb}}\tilde{x}_{\mathrm{b},i} + M_{\mathrm{cs}}\tilde{z}_{\mathrm{s},i} + M_{\mathrm{cd}}\tilde{x}_{\mathrm{d},i}$$

where $\tilde{z}_{\mathrm{s},i}$ is defined in (B.2), and M_{ca}^{-}, M_{cb}, M_{cs}, and M_{cd} are some constant matrices independent of ε. In view of the scalings (B.3), $B_{\mathrm{c}}F_{\mathrm{c}}\check{\tilde{x}}_{\mathrm{c},i}$ is modified to

$$B_{\mathrm{c}}F_{\mathrm{c}}\check{\tilde{x}}_{\mathrm{c},i} = M_{\mathrm{ca}}^{-}\tilde{x}_{\mathrm{a},i}^{-} + M_{\mathrm{cb}}\tilde{x}_{\mathrm{b},i} + \varepsilon^{-\bar{r}}M_{\mathrm{cs}}\tilde{z}_{\mathrm{s}\varepsilon,i} + \varepsilon^{-(\bar{r}-1)}M_{\mathrm{cd}\varepsilon}\tilde{x}_{\mathrm{d}\varepsilon,i}$$

where we have used the fact that $M_{\mathrm{cd}}\widetilde{S}^{-1} = \varepsilon^{-(\bar{r}-1)}M_{\mathrm{cd}\varepsilon}$ where $M_{\mathrm{cd}\varepsilon}$ is uniformly bounded for all $\varepsilon \in (0,1]$ since $\|M_{\mathrm{cd}}\widetilde{S}^{-1}\| = \mathscr{O}(\varepsilon^{-(\bar{r}-1)})$. Likewise, there exist some constant matrices $M_{j\mathrm{a}}^{-}$, $M_{j\mathrm{b}}$, $M_{j\mathrm{s}}$, and $M_{j\mathrm{d}}$ for $j \in \Omega$ that $M_{j\mathrm{d}}\widetilde{S}^{-1} = \varepsilon^{-(\bar{r}-1)}M_{j\mathrm{d}\varepsilon}$, where $M_{j\mathrm{d}\varepsilon}$ is uniformly bounded for all $\varepsilon \in (0,1]$, such that

$$B_{j\mathrm{d}}\widetilde{u}_{j1,i} = M_{j\mathrm{a}}^{-}\tilde{x}_{\mathrm{a},i}^{-} + M_{j\mathrm{b}}\tilde{x}_{\mathrm{b},i} + \varepsilon^{-\bar{r}}M_{j\mathrm{s}}\tilde{z}_{\mathrm{s}\varepsilon,i} + \varepsilon^{-(\bar{r}-1)}M_{j\mathrm{d}\varepsilon}\tilde{x}_{\mathrm{d}\varepsilon,i}$$

We also need to express $B_{j\mathrm{d}}\widetilde{u}_{j2,i}$ in terms of $\tilde{x}_i$ to be able to close the loop around agent i. For every $j \in \Omega$, partition $F_{j\mathrm{d}} = [F_{j1\mathrm{d}}, \cdots, F_{jj\mathrm{d}}]$ where $F_{jk\mathrm{d}} \in \mathbb{R}^{q_j \times q_j}$ for $k = 1, \cdots, j$. Then, one can show that $B_{j\mathrm{d}}\widetilde{u}_{j2,i}$ is equal to

$$\varepsilon^{-j}B_{j\mathrm{d}}F_{j\mathrm{d}}\left(C_{j\mathrm{d}}^{\mathrm{T}}F_{j\mathrm{a}}^{+}\check{\tilde{x}}_{\mathrm{a},i}^{+} + C_{j\mathrm{d}}^{\mathrm{T}}F_{j\mathrm{b}}\check{\tilde{x}}_{\mathrm{b},i}\right) + \varepsilon^{-j}B_{j\mathrm{d}}F_{j1\mathrm{d}}\check{\tilde{x}}_{j1\mathrm{d},i}$$
$$+ \varepsilon^{-j}B_{j\mathrm{d}}\textstyle\sum_{k=2}^{j}F_{jk\mathrm{d}}\varepsilon^{k-1}\check{\tilde{x}}_{jk\mathrm{d},i} \tag{B.6}$$

in which we have used the fact that $\check{\tilde{x}}_{j\mathrm{d},i} = \mathrm{col}\{\check{\tilde{x}}_{jk\mathrm{d},i}\}$, $\check{\tilde{x}}_{jk\mathrm{d},i} \in \mathbb{R}^{q_j}$, for all $k = 1, \cdots, j$ and every $j \in \Omega$. According to Assumptions 1-(4),(5), the first line of (B.6) depends only on $\tilde{x}_{\mathrm{a},i}^{-}$, $\tilde{x}_{\mathrm{b},i}$ and $\tilde{x}_{\mathrm{d},i}$. Therefore, there exist some constant matrices $N_{j\mathrm{a}}^{-}$, $N_{j\mathrm{b}}$, and $N_{j\mathrm{d}}$ for each $j \in \Omega$, independent of ε such that the first line of (B.6) is described by $\varepsilon^{-j}(N_{j\mathrm{a}}^{-}\tilde{x}_{\mathrm{a},i}^{-} + N_{j\mathrm{b}}\tilde{x}_{\mathrm{b},i} + \varepsilon^{-(\bar{r}-1)}N_{j\mathrm{d}\varepsilon}\tilde{x}_{\mathrm{d}\varepsilon,i})$ where $N_{j\mathrm{d}}\widetilde{S}^{-1} = \varepsilon^{-(\bar{r}-1)}N_{j\mathrm{d}\varepsilon}$, where $N_{j\mathrm{d}\varepsilon}$ is uniformly bounded for all $\varepsilon \in (0,1]$. Similarly, $\check{\tilde{x}}_{jk\mathrm{d},i}$ for $k = 2, \cdots, j$ and $j \in \Omega$ can be expressed as a linear combination of components of $\tilde{x}_i$. Thus, we get

$$B_{j\mathrm{d}}\textstyle\sum_{k=2}^{j}F_{jk\mathrm{d}}\varepsilon^{k-1}\check{\tilde{x}}_{jk\mathrm{d},i} = \varepsilon\textstyle\sum_{k=2}^{j}\varepsilon^{k-2}(\check{M}_{jk\mathrm{a}}^{-}\tilde{x}_{\mathrm{a},i}^{-} + \check{M}_{jk\mathrm{b}}\tilde{x}_{\mathrm{b},i}$$
$$+ \varepsilon^{-\bar{r}}\check{M}_{jk\mathrm{s}}\tilde{z}_{\mathrm{s}\varepsilon,i} + \varepsilon^{-(\bar{r}-1)}\check{M}_{jk\varepsilon}\tilde{x}_{\mathrm{d}\varepsilon,i})$$

It is straightforward to verify that $\check{M}_{jk\varepsilon}$, $k = 2, \cdots, j$ and $j \in \Omega$, is uniformly bounded for all $\varepsilon \in (0,1]$. Denoting

$$\check{M}_{j\mathrm{d}} = \textstyle\sum_{k=2}^{j}\varepsilon^{k-2}\check{M}_{jk\varepsilon}, \qquad \check{M}_{j\mathrm{s}} = \textstyle\sum_{k=2}^{j}\varepsilon^{k-2}\check{M}_{jk\mathrm{s}}$$
$$\check{M}_{j\mathrm{a}}^{-} = \textstyle\sum_{k=2}^{j}\varepsilon^{k-2}\check{M}_{jk\mathrm{a}}^{-}, \qquad \check{M}_{j\mathrm{b}} = \textstyle\sum_{k=2}^{j}\varepsilon^{k-2}\check{M}_{jk\mathrm{b}}$$

which are all uniformly bounded in ε. Thus, it turns out that $B_{j\mathrm{d}}\sum_{k=2}^{j}F_{jk\mathrm{d}}\varepsilon^{k-1}\check{\tilde{x}}_{jk\mathrm{d},i} = \varepsilon\check{M}_{j\mathrm{a}}^{-}\tilde{x}_{\mathrm{a},i}^{-} + \varepsilon\check{M}_{j\mathrm{b}}\tilde{x}_{\mathrm{b},i} + \varepsilon^{-(\bar{r}-1)}\check{M}_{j\mathrm{s}}\tilde{z}_{\mathrm{s}\varepsilon,i} + \varepsilon^{-(\bar{r}-2)}\check{M}_{j\mathrm{d}}\tilde{x}_{\mathrm{d}\varepsilon,i}$. Now, we are ready to find the closed-loop equations. Considering (9), we introduce the state transformations:

$$x_{\mathrm{a}\varepsilon,i}^{-} = \varepsilon x_{\mathrm{a},i}^{-}, \qquad x_{\mathrm{c}\varepsilon,i} = \varepsilon x_{\mathrm{c},i}, \qquad z_{j\mathrm{d}\varepsilon,i} = \check{S}_j z_{j\mathrm{d},i} \tag{B.7}$$

Denote $z_{\mathrm{d}\varepsilon,i} = \mathrm{col}\{z_{j\mathrm{d}\varepsilon,i}\}\ \forall j \in \Omega$. In light of Step 2, one may demonstrate the dynamics of the systems as

$$\dot{x}_{\mathrm{a}\varepsilon,i}^{-} = A_{\mathrm{a}}^{-}x_{\mathrm{a}\varepsilon,i}^{-} + \varepsilon L_{\mathrm{as}}^{-}x_{\mathrm{s},i} + \varepsilon L_{\mathrm{ad}}^{-}C_{\mathrm{d}}z_{\mathrm{d}\varepsilon,i} + \varepsilon G_{\mathrm{a}}^{-}\mathbf{w}_i$$
$$\dot{x}_{\mathrm{s},i} = A_{\mathrm{ss}}x_{\mathrm{s},i} + L_{\mathrm{sd}}C_{\mathrm{d}}z_{\mathrm{d}\varepsilon,i}$$
$$\dot{x}_{\mathrm{c}\varepsilon,i} = A_{\mathrm{cc}}x_{\mathrm{c}\varepsilon,i} + B_{\mathrm{c}}E_{\mathrm{ca}}^{-}x_{\mathrm{a}\varepsilon,i}^{-} + \varepsilon E_{\mathrm{cs}}x_{\mathrm{s},i} + \varepsilon L_{\mathrm{cd}}C_{\mathrm{d}}z_{\mathrm{d}\varepsilon,i}$$
$$+ \varepsilon G_{\mathrm{c}}\mathbf{w}_i + \varepsilon M_{\mathrm{ca}}^{-}\tilde{x}_{\mathrm{a},i}^{-} + \varepsilon M_{\mathrm{cb}}\tilde{x}_{\mathrm{b},i}$$
$$+ \varepsilon^{-(\bar{r}-1)}(M_{\mathrm{cs}}\tilde{z}_{\mathrm{s}\varepsilon,i} + \varepsilon M_{\mathrm{cd}\varepsilon}\tilde{x}_{\mathrm{d}\varepsilon,i})$$
$$\varepsilon\dot{z}_{\mathrm{d}\varepsilon,i} = A_{\mathrm{dd}}z_{\mathrm{d}\varepsilon,i} + \varepsilon L_{\mathrm{dd}}z_{\mathrm{d}\varepsilon,i} + \varepsilon L_{\mathrm{ds}}x_{\mathrm{s},i} + \varepsilon G_{\mathrm{dd}}\mathbf{w}_i$$
$$+ \varepsilon M_{\mathrm{da}}^{-}\tilde{x}_{\mathrm{a},i}^{-} + \varepsilon M_{\mathrm{db}}\tilde{x}_{\mathrm{b},i} + \varepsilon^{-(\bar{r}-1)}M_{\mathrm{ds}}\tilde{z}_{\mathrm{s}\varepsilon,i}$$
$$+ \varepsilon^{-(\bar{r}-2)}M_{\mathrm{dd}}\tilde{x}_{\mathrm{d}\varepsilon,i} + \varepsilon^{-(\bar{r}-1)}N_{\mathrm{dd}}\tilde{x}_{\mathrm{d}\varepsilon,i}$$
$$+ N_{\mathrm{da}}^{-}\tilde{x}_{\mathrm{a},i}^{-} + N_{\mathrm{db}}\tilde{x}_{\mathrm{b},i}$$

where $L_{\mathrm{as}}^{-} = -L_{\mathrm{ad}}^{-}F_{\mathrm{s}} + [0,\quad L_{\mathrm{ab}}^{-}C_{\mathrm{b}}]$ and $E_{\mathrm{cs}} = -L_{\mathrm{cd}}F_{\mathrm{s}} + [B_{\mathrm{c}}E_{\mathrm{ca}}^{+},\quad L_{\mathrm{cb}}C_{\mathrm{b}}]$. Also, for $j \in \Omega$, we have defined

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$L^*_{jd} = C^{\mathrm{T}}_{jd} F_{js} L_{\mathrm{sd}} + L_{jd}$, $L_{js} = C^{\mathrm{T}}_{jd} F_{js} A_{\mathrm{ss}} - L_{jd} F_{\mathrm{s}}$, and

$$\begin{aligned}
&A_{\mathrm{dd}} = \mathrm{diag}\{A^*_{jd}\} && G_{\mathrm{dd}} = \mathrm{stack}\{\check{S}_j G_{jd}\} \\
&L_{\mathrm{dd}} = \mathrm{stack}\{\check{S}_j L^*_{jd}\} C_{\mathrm{d}} && L_{\mathrm{ds}} = \mathrm{stack}\{\check{S}_j L_{js}\} \\
&M^-_{\mathrm{da}} = \mathrm{stack}\{\varepsilon^{j-1} M^-_{ja} + \check{M}^-_{ja}\} && \\
&M_{\mathrm{db}} = \mathrm{stack}\{\varepsilon^{j-1} M_{jb} + \check{M}_{jb}\} && N^-_{\mathrm{da}} = \mathrm{stack}\{N^-_{ja}\} \\
&M_{\mathrm{ds}} = \mathrm{stack}\{\varepsilon^{j-1} M_{js} + \check{M}_{js}\} && N_{\mathrm{db}} = \mathrm{stack}\{N_{jb}\} \\
&M_{\mathrm{dd}} = \mathrm{stack}\{\varepsilon^{j-1} M_{jd\varepsilon} + \check{M}_{jd}\} && N_{\mathrm{dd}} = \mathrm{stack}\{N_{jd\varepsilon}\}
\end{aligned}$$

Closed-loop Equations for Agent i - Compact Form: Define

$$\begin{aligned}
&\tilde{\mathbf{z}}_i = \mathrm{col}\{\tilde{x}^-_{\mathrm{a},i}, \tilde{x}_{\mathrm{b},i}, \tilde{x}_{\mathrm{d}\varepsilon,i}, \tilde{z}_{\mathrm{s}\varepsilon,i}\}, && \mathbf{z}_i = \mathrm{col}\{x_{\mathrm{c}\varepsilon,i}, x_{\mathrm{s},i}, x^-_{\mathrm{a}\varepsilon,i}, z_{\mathrm{d}\varepsilon,i}\} \\
&\bar{\mathscr{A}} = \mathrm{diag}\{\bar{A}^-_{\mathrm{a}}, \bar{A}_{\mathrm{b}}, \bar{\mathbb{A}}_{\mathrm{dd}}, \bar{A}_{\mathrm{ss}}\}, && \mathscr{A} = \mathrm{diag}\{A_{\mathrm{cc}}, A_{\mathrm{ss}}, A^-_{\mathrm{a}}, A_{\mathrm{dd}}\} \\
&\bar{\mathscr{S}} = \mathrm{diag}\{\mathrm{I}, \mathrm{I}, \bar{\mathbb{S}}, \mathrm{I}\}, && \mathscr{S} = \mathrm{diag}\{\mathrm{I}, \mathrm{I}, \mathrm{I}, \varepsilon \mathrm{I}\}
\end{aligned}$$

Now, the closed-loop equations are recast as

$$\mathscr{S}\,\dot{\mathbf{z}}_i = \mathscr{A}\,\mathbf{z}_i + \mathscr{L}\,\mathbf{z}_i + \varepsilon\,\mathscr{E}\,\mathbf{w}_i + \mathscr{D}\,\tilde{\mathbf{z}}_i \tag{B.8a}$$
$$\bar{\mathscr{S}}\,\dot{\tilde{\mathbf{z}}}_i = \bar{\mathscr{A}}\,\tilde{\mathbf{z}}_i + \bar{\mathscr{L}}\,\tilde{\mathbf{z}}_i + \varepsilon^{\bar{\mathsf{r}}}\bar{\mathscr{E}}\,\mathbf{w}_i - \textstyle\sum_{k=1}^N g_{ik} \bar{\mathscr{D}}\,\tilde{\mathbf{z}}_k \tag{B.8b}$$

where $\mathscr{L} = \varepsilon \mathscr{L}_\varepsilon + \mathscr{L}_0$, $\|\varepsilon \mathscr{L}_\varepsilon\| = \mathscr{O}(\varepsilon)$ and $\bar{\mathscr{L}} = \varepsilon \bar{\mathscr{L}}_\varepsilon + \bar{\mathscr{L}}_0$, $\|\varepsilon \bar{\mathscr{L}}_\varepsilon\| = \mathscr{O}(\varepsilon)$; also $\mathscr{D} = \mathscr{D}_\varepsilon + \varepsilon^{-(\bar{\mathsf{r}}-1)} \mathscr{D}_0$. The norms of $\mathscr{L}_0$, $\bar{\mathscr{L}}_0$, $\mathscr{D}_0$, $\bar{\mathscr{D}}$, $\mathscr{E}$, and $\bar{\mathscr{E}}$ are uniformly bounded for all ε.

$$\mathscr{L}_\varepsilon = \begin{bmatrix} 0 & E_{\mathrm{cs}} & 0 & L_{\mathrm{cd}} C_{\mathrm{d}} \\ 0 & 0 & 0 & 0 \\ 0 & L^-_{\mathrm{as}} & 0 & L^-_{\mathrm{ad}} C_{\mathrm{d}} \\ 0 & L_{\mathrm{ds}} & 0 & L_{\mathrm{dd}} \end{bmatrix}, \qquad \mathscr{L}_0 = \begin{bmatrix} 0 & 0 & B_{\mathrm{c}} E^-_{\mathrm{ca}} & 0 \\ 0 & 0 & 0 & L_{\mathrm{sd}} C_{\mathrm{d}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\bar{\mathscr{L}}_\varepsilon = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \varepsilon^{\bar{\mathsf{r}}-1} \bar{\mathbb{E}}^-_{\mathrm{da}} & \varepsilon^{\bar{\mathsf{r}}-1} \bar{\mathbb{E}}_{\mathrm{db}} & \bar{\mathbb{E}}_{\mathrm{dd}\varepsilon} & 0 \\ \varepsilon^{\bar{\mathsf{r}}-1} \bar{E}^-_{\mathrm{sa}} & \varepsilon^{\bar{\mathsf{r}}-1} \bar{E}_{\mathrm{sb}} & \bar{\mathbb{E}}_{\mathrm{sd}\varepsilon} & 0 \end{bmatrix}, \qquad \bar{\mathscr{E}} = \begin{bmatrix} 0 \\ 0 \\ \bar{\mathbb{G}}_{\mathrm{dd}} \\ \bar{G}_{\mathrm{ss}} \end{bmatrix}$$

$$\bar{\mathscr{L}}_0 = \begin{bmatrix} 0 & \bar{L}^-_{\mathrm{ab}} \bar{C}_{\mathrm{b}} & \bar{L}^-_{\mathrm{ad}} \bar{C}_{\mathrm{d}} & 0 \\ 0 & 0 & \bar{L}_{\mathrm{bd}} \bar{C}_{\mathrm{d}} & 0 \\ 0 & 0 & 0 & \bar{\mathbb{E}}_{\mathrm{ds}} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathscr{E} = \begin{bmatrix} G_{\mathrm{c}} \\ 0 \\ G^-_{\mathrm{a}} \\ G_{\mathrm{dd}} \end{bmatrix}$$

$$\bar{\mathscr{D}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \bar{K}_{\mathrm{b}} \bar{C}_{\mathrm{b}} & 0 & 0 \\ 0 & 0 & \bar{\mathbb{K}}_{\mathrm{dd}} \bar{C}_{\mathrm{d}} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathscr{D}_0 = \begin{bmatrix} 0 & 0 & 0 & M_{\mathrm{cs}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & N_{\mathrm{dd}} & M_{\mathrm{ds}} \end{bmatrix}$$

$$\mathscr{D}_\varepsilon = \begin{bmatrix} \varepsilon M^-_{\mathrm{ca}} & \varepsilon M_{\mathrm{cb}} & \varepsilon^{-(\bar{\mathsf{r}}-2)} M_{\mathrm{cd}\varepsilon} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \varepsilon M^-_{\mathrm{da}} + N^-_{\mathrm{da}} & \varepsilon M_{\mathrm{db}} + N_{\mathrm{db}} & \varepsilon^{-(\bar{\mathsf{r}}-2)} M_{\mathrm{dd}} & 0 \end{bmatrix}$$

Closed-loop Equations for the Multi-agent System: Collect the states as $\chi = \mathrm{col}\{\mathbf{z}_i\}$, $\tilde{\chi} = \mathrm{col}\{\tilde{\mathbf{z}}_i\}$ for all $i \in \mathfrak{S}$. Then, the collective dynamics are described by

$$(\mathrm{I}_N \otimes \mathscr{S})\dot{\chi} = (\mathrm{I}_N \otimes \mathscr{A})\,\chi + (\mathrm{I}_N \otimes \mathscr{L})\chi + \varepsilon(\mathrm{I}_N \otimes \mathscr{E})w + (\mathrm{I}_N \otimes \mathscr{D})\tilde{\chi} \tag{B.9a}$$
$$(\mathrm{I}_N \otimes \bar{\mathscr{S}})\dot{\tilde{\chi}} = \left((\mathrm{I}_N \otimes \bar{\mathscr{A}}) - (\mathsf{G} \otimes \bar{\mathscr{D}})\right)\tilde{\chi} + (\mathrm{I}_N \otimes \bar{\mathscr{L}})\tilde{\chi} + \varepsilon^{\bar{\mathsf{r}}}(\mathrm{I}_N \otimes \bar{\mathscr{E}})w \tag{B.9b}$$

Recall $y_{\mathrm{d},i} = \mathrm{col}\{y_{jd,i}\}$ for all $j \in \Omega$ and every agent $i \in \mathfrak{S}$. Thus, we obtain $y_{jd,i} = C_{jd} z_{jd\varepsilon,i} - F_{js} x_{\mathrm{s},i}$. In addition, $y_{\mathrm{b},i} = [0,\ C_{\mathrm{b}}] x_{\mathrm{s},i}$. It implies there exists a matrix Γ^*_{y}, independent of ε, such that $\mathbf{y}_i = \Gamma_{\mathrm{y}} \Gamma^*_{\mathrm{y}} \mathbf{z}_i$. Therefore, in view of (2), $\zeta = (\mathsf{G} \otimes \Gamma_{\mathrm{y}} \Gamma^*_{\mathrm{y}})\chi$.

Let $\mathbf{1}, \mathbf{1}_L \in \mathbb{R}^N$: $\mathsf{G}\mathbf{1} = 0$ and $\mathbf{1}_L^{\mathrm{T}} \mathsf{G} = 0$. Suppose the Jordan form of G is obtained using the matrix U which is chosen as $U = [\bar{U}, \mathbf{1}] \Rightarrow (U^{-1})^{\mathrm{T}} = [\bar{U}_L, \mathbf{1}_L]$. Thus, one can find the Jordan form as $U^{-1} \mathsf{G} U = \mathrm{diag}\{\Delta, 0\}$. It implies that $\mathsf{G}U = [\check{\mathsf{G}},\ 0]$ where $\check{\mathsf{G}} = \bar{U}\Delta$. We introduce the following state transformations

$$\begin{bmatrix} e \\ e_0 \end{bmatrix} = (U^{-1} \otimes \mathrm{I}_n)\,\chi, \qquad \begin{bmatrix} \tilde{e} \\ \tilde{e}_0 \end{bmatrix} = (U^{-1} \otimes \mathrm{I}_n)\,\tilde{\chi} \tag{B.10}$$

where $e_0, \tilde{e}_0 \in \mathbb{R}^n$. Denote $\bar{N} = N - 1$. Then, we find two sets of equations. The first set is given as bellow.

$$(\mathrm{I}_{\bar{N}} \otimes \mathscr{S})\dot{e} = (\mathrm{I}_{\bar{N}} \otimes (\mathscr{A} + \mathscr{L}_0))\,e + \varepsilon(\mathrm{I}_{\bar{N}} \otimes \mathscr{L}_\varepsilon)e + \varepsilon^{-(\bar{\mathsf{r}}-1)}\left(\mathrm{I}_{\bar{N}} \otimes (\varepsilon^{\bar{\mathsf{r}}-1}\mathscr{D}_\varepsilon + \mathscr{D}_0)\right)\tilde{e} + \varepsilon(\bar{U}_L^{\mathrm{T}} \otimes \mathscr{E})w \tag{B.11a}$$
$$(\mathrm{I}_{\bar{N}} \otimes \bar{\mathscr{S}})\dot{\tilde{e}} = \left(\mathrm{I}_{\bar{N}} \otimes (\bar{\mathscr{A}} + \bar{\mathscr{L}}_0) - \Delta \otimes \bar{\mathscr{D}}\right)\tilde{e} + \varepsilon(\mathrm{I}_{\bar{N}} \otimes \bar{\mathscr{L}}_\varepsilon)\tilde{e} + \varepsilon^{\bar{\mathsf{r}}}(\bar{U}_L^{\mathrm{T}} \otimes \bar{\mathscr{E}})w \tag{B.11b}$$

and $\zeta = (\check{\mathsf{G}} \otimes \Gamma_{\mathrm{y}} \Gamma^*_{\mathrm{y}})e$. The state $(e_0, \tilde{e}_0)$ determines the agreement trajectories when $\zeta = 0$.

H_∞ Analysis: Consider the reduced-order system (B.11) with the controlled output ζ. Choose $\rho > 0$ such that $\zeta^{\mathrm{T}}\zeta \leq \rho^2 e^{\mathrm{T}} e$ for all $\varepsilon \in (0, 1]$.

The matrix $\mathscr{A}$ is Hurwitz stable because A^-_{a}, A_{ss}, A_{cc}, and A^*_{jd}, $\forall j \in \Omega$, are Hurwitz stable. Due to the upper block-triangular structure of $\mathscr{L}_0$ where the blocks along the diagonal are zero, the matrix $(\mathrm{I}_{\bar{N}} \otimes (\mathscr{A} + \mathscr{L}_0))$ is upper block-

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triangular and Hurwitz stable. Therefore, there exists a symmetric $\mathscr{Q} > (\rho^2+4)\mathrm{I}_n$ such that the Lyapunov equation:

$$(\mathscr{A}+\mathscr{L}_0)^{\mathrm{T}}\mathscr{P}+\mathscr{P}(\mathscr{A}+\mathscr{L}_0) = -\mathscr{Q}$$

has a unique positive definite and symmetric solution $\mathscr{P}$ which is block-diagonal with the block sizes that correspond to the block sizes in $\mathscr{S}$. It guarantees that $\mathscr{P}$ and $\mathscr{S}$ commute. Choose $V_c = e^{\mathrm{T}}(\mathrm{I}_{\bar{N}}\otimes\mathscr{S}\mathscr{P})e$, and differentiate it along the trajectories of (B.11a); it gives rise to

$$\begin{aligned}\dot{V}_c \le & -\zeta^{\mathrm{T}}\zeta - 3\|e\|^2 - (1-2\varepsilon\mu_\varepsilon)\|e\|^2 \\ & + 2\varepsilon^{-(\bar{\mathrm{r}}-1)}\mu_{\mathrm{d}\varepsilon}\|e\|\|\tilde{e}\| + 2\varepsilon\mu_{\mathrm{w}}\|e\|\|w\|\end{aligned}$$

where $\mu_\varepsilon \ge \max_{\varepsilon\in(0,1]}\|\mathscr{P}\mathscr{L}_\varepsilon\|$, $\mu_{\mathrm{w}} \ge \|(\bar{U}_L^{\mathrm{T}}\otimes\mathscr{P}\mathscr{E})\|$, and $\mu_{\mathrm{d}\varepsilon} \ge \max_{\varepsilon\in(0,1]}\|\mathscr{P}(\varepsilon^{\bar{\mathrm{r}}-1}\mathscr{D}_\varepsilon+\mathscr{D}_0)\|$. Since μ_ε is bounded, there exists a sufficiently small $\varepsilon_{11}\in(0,1]$ such that $1-2\varepsilon\mu_\varepsilon > 0$ for every $\varepsilon\in(0,\varepsilon_{11}]$. Therefore,

$$\dot{V}_c \le -\zeta^{\mathrm{T}}\zeta - 3\|e\|^2 + 2\varepsilon^{-(\bar{\mathrm{r}}-1)}\mu_{\mathrm{d}\varepsilon}\|e\|\|\tilde{e}\| + 2\varepsilon\mu_{\mathrm{w}}\|e\|\|w\|$$

In (B.11b), let $\bar{\mathscr{A}}^* = \big((\mathrm{I}_{\bar{N}}\otimes(\bar{\mathscr{A}}+\bar{\mathscr{L}}_0) - (\Delta\otimes\bar{\mathscr{D}})\big)$. From the structure of Δ, it is observed that $\bar{\mathscr{A}}^*$ is upper block-triangular. Thus, $\bar{\mathscr{A}}^*$ is Hurwitz stable iff all matrices on the main diagonal are Hurwitz stable. In other words, $\bar{\mathscr{A}}+\bar{\mathscr{L}}_0-\lambda\bar{\mathscr{D}}$ must be Hurwitz stable for all λ's which are nonzero eigenvalues of the Laplacian matrix G. Notice that since $\mathscr{G}\in\mathscr{G}_\beta$, $\mathrm{Re}(\lambda)>\beta>0$ if $\lambda\neq 0$. Since $\bar{\mathscr{A}}-\lambda\bar{\mathscr{D}}$ is a block-diagonal matrix and $\bar{\mathscr{L}}_0$ is upper block-triangular where the blocks along the diagonal are zero, the eigenvalues of $\bar{\mathscr{A}}^*$ are determined by the eigenvalues of $\bar{A}_{\mathrm{a}}^-, \bar{A}_{\mathrm{ss}}, \bar{A}_{\mathrm{bb}} = \bar{A}_{\mathrm{b}} - \lambda\bar{K}_{\mathrm{b}}\bar{C}_{\mathrm{b}}$, and $\bar{A}_{j\mathrm{d}}^* = \bar{A}_{j\mathrm{d}} - \lambda\bar{K}_{j\mathrm{d}}\bar{C}_{j\mathrm{d}}$.

$\bar{A}_{\mathrm{a}}^-$ is Hurwitz stable by definition, and $\bar{A}_{\mathrm{ss}}$ was made Hurwitz stable in Step 6. According to Step 6, it can be confirmed that $\bar{A}_{\mathrm{bb}}$ and $\bar{A}_{j\mathrm{d}}$, $\forall j\in\bar{\Omega}$, are Hurwitz stable. To see that we recall $\bar{K}_{\mathrm{b}} = \bar{P}_{\mathrm{b}}\bar{C}_{\mathrm{b}}^{\mathrm{T}}$ and $\beta\ge\tau$; therefore, we can show

$$\begin{aligned}\bar{A}_{\mathrm{bb}}\bar{P}_{\mathrm{b}} + \bar{P}_{\mathrm{b}}\bar{A}_{\mathrm{bb}}^{\mathrm{H}} &= \bar{A}_{\mathrm{b}}\bar{P}_{\mathrm{b}} + \bar{P}_{\mathrm{b}}\bar{A}_{\mathrm{b}}^{\mathrm{T}} - 2\mathrm{Re}(\lambda)\bar{P}_{\mathrm{b}}\bar{C}_{\mathrm{b}}^{\mathrm{T}}\bar{C}_{\mathrm{b}}\bar{P}_{\mathrm{b}} \\ &= \bar{A}_{\mathrm{b}}\bar{P}_{\mathrm{b}} + \bar{P}_{\mathrm{b}}\bar{A}_{\mathrm{b}}^{\mathrm{T}} - 2\tau\bar{P}_{\mathrm{b}}\bar{C}_{\mathrm{b}}^{\mathrm{T}}\bar{C}_{\mathrm{b}}\bar{P}_{\mathrm{b}} \\ &\quad - 2(\mathrm{Re}(\lambda)-\tau)\bar{P}_{\mathrm{b}}\bar{C}_{\mathrm{b}}^{\mathrm{T}}\bar{C}_{\mathrm{b}}\bar{P}_{\mathrm{b}} \le -\mathrm{I}_{\bar{n}_{\mathrm{b}}}\end{aligned}$$

It follows that $\bar{A}_{\mathrm{bb}}$ is Hurwitz stable. Similarly, it is confirmed that $\bar{A}_{j\mathrm{d}}^*$'s are Hurwitz stable. Hence, $\bar{\mathscr{A}}+\bar{\mathscr{L}}_0-\lambda_i\bar{\mathscr{D}}$ and $\bar{\mathscr{A}}^*$ are Hurwitz stable for every nonzero λ_i. Accordingly, for λ_i, $i=1,\cdots,N-1$, there exists a symmetric $\tilde{\mathrm{Q}}_i>0$ such that the Lyapunov equation

$$(\bar{\mathscr{A}}+\bar{\mathscr{L}}_0-\lambda_i\bar{\mathscr{D}})^{\mathrm{H}}\tilde{\mathrm{P}}_i + \tilde{\mathrm{P}}_i(\bar{\mathscr{A}}+\bar{\mathscr{L}}_0-\lambda_i\bar{\mathscr{D}}) = -\tilde{\mathrm{Q}}_i$$

has a unique solution $\tilde{\mathrm{P}}_i = \tilde{\mathrm{P}}_i^{\mathrm{T}} > 0$ which is block-diagonal with the block sizes that correspond to the block sizes in $\bar{\mathscr{S}}$. Let $\tilde{\mathrm{q}}_i>0$ be such that $\tilde{\mathrm{q}}_i\mathrm{I}\le\tilde{\mathrm{Q}}_i$, and $\tilde{\eta}_i = \|\tilde{\mathrm{P}}_i\bar{\mathscr{D}}\|$. Following the proof of Proposition 1 in (Peymani et al., 2014), we can show that the block diagonal matrix $\tilde{\mathscr{P}}$ constructed as

$$\tilde{\mathscr{P}} = \mathrm{diag}\{\delta_1\tilde{\mathrm{P}}_1,\cdots,\delta_{N-1}\tilde{\mathrm{P}}_{N-1}\} \tag{B.12}$$

where $\delta_{N-1}=1$ and $\delta_i = \delta_{i+1}\tilde{\mathrm{q}}_i\tilde{\mathrm{q}}_{i+1}/9\tilde{\eta}_i^2$ for $i=1,\cdots,N-2$ (implying $\|\tilde{\mathscr{P}}\|$ is bounded for any $\beta>0$) solves the Lyapunov function $(\bar{\mathscr{A}}^*)^{\mathrm{H}}\tilde{\mathscr{P}}+\tilde{\mathscr{P}}\bar{\mathscr{A}}^* = -\tilde{\mathscr{Q}}$ for some symmetric $\tilde{\mathscr{Q}} > (3+\mu_{\mathrm{d}\varepsilon}^2)\mathrm{I}_{\bar{N}n}$. Considering that $\tilde{\mathscr{P}}$ and $(\mathrm{I}_{\bar{N}}\otimes\bar{\mathscr{S}})$ commute, we choose $V_o = \tilde{e}^{\mathrm{T}}(\mathrm{I}_{\bar{N}}\otimes\bar{\mathscr{S}})\tilde{\mathscr{P}}\tilde{e}$, and take derivative

$$\dot{V}_o \le -(2+\mu_{\mathrm{d}\varepsilon}^2)\|\tilde{e}\|^2 - (1-2\varepsilon\tilde{\rho}_\varepsilon)\|\tilde{e}\|^2 + 2\varepsilon^{\bar{\mathrm{r}}}\tilde{\rho}_{\mathrm{w}}\|\tilde{e}\|\|w\|$$

where $\tilde{\rho}_\varepsilon \ge \max_{\varepsilon\in(0,1]}\|\tilde{\mathscr{P}}(\mathrm{I}_{\bar{N}}\otimes\bar{\mathscr{L}}_\varepsilon)\|$ and $\tilde{\rho}_{\mathrm{w}} \ge \|\tilde{\mathscr{P}}(\bar{U}_L^{\mathrm{T}}\otimes\bar{\mathscr{E}})\|$. Because $\tilde{\mathscr{P}}$ is bounded for all $\beta>0$ (i.e. for all network graphs $\mathscr{G}\in\mathscr{G}_\beta$), $\tilde{\rho}_\varepsilon$ and $\tilde{\rho}_{\mathrm{w}}$ are bounded for all $\beta>0$. Accordingly, there exists an $\varepsilon_{22}\in(0,1]$ such that $1-2\varepsilon\tilde{\rho}_\varepsilon>0$ for every $\varepsilon\in(0,\varepsilon_{22}]$; thus, for every $\varepsilon\in(0,\varepsilon_{22}]$, we obtain

$$\dot{V}_o \le -(2+\mu_{\mathrm{d}\varepsilon}^2)\|\tilde{e}\|^2 + 2\varepsilon^{\bar{\mathrm{r}}}\tilde{\rho}_{\mathrm{w}}\|\tilde{e}\|\|w\|$$

Choose $V = V_c + \varepsilon^{-2(\bar{\mathrm{r}}-1)}V_o$. Let $\varepsilon_1^* = \min\{\varepsilon_{11},\varepsilon_{22}\}$. For every $\varepsilon\in(0,\varepsilon_1^*]$, an upper bound on $\dot{V}$ is given by

$$\begin{aligned}\dot{V} \le & -\zeta^{\mathrm{T}}\zeta - 3\|e\|^2 + 2\varepsilon^{-(\bar{\mathrm{r}}-1)}\mu_{\mathrm{d}\varepsilon}\|e\|\|\tilde{e}\| \\ & + 2\varepsilon\mu_{\mathrm{w}}\|e\|\|w\| \\ & - \varepsilon^{-2(\bar{\mathrm{r}}-1)}(2+\mu_{\mathrm{d}\varepsilon}^2)\|\tilde{e}\|^2 + 2\varepsilon^{-(\bar{\mathrm{r}}-2)}\tilde{\rho}_{\mathrm{w}}\|\tilde{e}\|\|w\| \\ \le & -\zeta^{\mathrm{T}}\zeta - 2\|e\|^2 - 2\varepsilon^{-2(\bar{\mathrm{r}}-1)}\|\tilde{e}\|^2 \\ & + 2\varepsilon(\mu_{\mathrm{w}}\|e\| + \varepsilon^{-(\bar{\mathrm{r}}-1)}\tilde{\rho}_{\mathrm{w}}\|\tilde{e}\|)\|w\| \\ & - \|e\|^2 + 2\varepsilon^{-(\bar{\mathrm{r}}-1)}\mu_{\mathrm{d}\varepsilon}\|e\|\|\tilde{e}\| - \varepsilon^{-2(\bar{\mathrm{r}}-1)}\mu_{\mathrm{d}\varepsilon}^2\|\tilde{e}\|^2\end{aligned}$$

The third line is equal to $-(\|e\| - \varepsilon^{-(\bar{\mathrm{r}}-1)}\mu_{\mathrm{d}\varepsilon}\|\tilde{e}\|)^2 \le 0$. Denote $\sigma_{\mathrm{w}} = \sqrt{2}\max\{\mu_{\mathrm{w}},\tilde{\rho}_{\mathrm{w}}\}$. Then, one may write

$$\begin{aligned}\dot{V} \le & -\zeta^{\mathrm{T}}\zeta - 2\|e\|^2 - 2\varepsilon^{-2(\bar{\mathrm{r}}-1)}\|\tilde{e}\|^2 \\ & + 2\varepsilon\sigma_{\mathrm{w}}\sqrt{\|e\|^2 + \varepsilon^{-2(\bar{\mathrm{r}}-1)}\|\tilde{e}\|^2}\,\|w\|\end{aligned}$$

where we have used the fact that $\|x\|+\|y\| \le \sqrt{2}\sqrt{x^2+y^2}$. Completing the square results in

$$\dot{V} \le -\zeta^{\mathrm{T}}\zeta - \|e\|^2 - \varepsilon^{-2(\bar{\mathrm{r}}-1)}\|\tilde{e}\|^2 + (\varepsilon\sigma_{\mathrm{w}})^2\|w\|^2$$

Hence, from the Kalman-Yakubovich-Popov Lemma, it follows that $\|T_{w\zeta}\|_\infty \le \varepsilon\sigma_{\mathrm{w}}$. We need to show that the impact of w on every mutual disagreement $\mathfrak{e}_{i,j}$ can be made arbitrarily small. We define $\mathfrak{e}_{i,j} = T_{w\mathfrak{e}}^{i,j}(s)w$, $i,j\in\mathfrak{S}, i>j$.

From (Peymani et al., 2014, Lemma 3), it follows that there exists $\hat{\sigma}>0$ such that $\|T_{w\mathfrak{e}}^{i,j}\|_\infty < \varepsilon\hat{\sigma}$. Therefore, for any given $\gamma>0$, there exists an $\varepsilon_2^*\in(0,\varepsilon_1^*]$ such that every $\varepsilon\in(0,\varepsilon_2^*]$ yields $\|T_{w\mathfrak{e}}^{i,j}\|_\infty < \gamma$.

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